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Topological Amplitudes in Heterotic Superstring Theory*

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Abstract

We show that certain heterotic string amplitudes are given in terms of correlators of the twisted topological (2,0) SCFT, corresponding to the internal sector of the $N = 1$ spacetime supersymmetric background. The genus g topological partition function F^g corresponds to a term in the effective action of the form W^{2g}, where W is the gauge or gravitational superfield. We study also recursion relations related to holomorphic anomalies, showing that, contrary to the type II case, they involve correlators of anti-chiral superfields. The corresponding terms in the effective action are of the form $W^{2g}\Pi^n$, where Π is a chiral superfield obtained by chiral projection of a general superfield. We observe that the structure of the recursion relations is that of $N = 1$ spacetime supersymmetry Ward identity. We give also a solution of the tree level recursion relations and discuss orbifold examples.

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1. Introduction

The low energy behavior of superstring theory is described by a locally supersymmetric effective field theory of the massless modes. The structure of the effective theory has been studied most extensively in the leading low-energy approximation, the so-called two derivative level, at which the theory can be described by the standard supergravity lagrangians. One of the most interesting aspects of superstrings is the relation between their low-energy interactions and the topological properties of the underlying world-sheet $N = 2$ superconformal field theory.

In type II superstring Calabi-Yau compactifications, special geometry of $N = 2$ supergravity can be described in the framework of a topological field theory of the twisted Calabi-Yau sigma-model [1]. The topological partition function of such a model can be defined on a Riemann surface of arbitrary genus g. At genus one, it represents a new supersymmetric index [2] which determines the one-loop corrections to the four-derivative gravitational R^2 couplings [3]. At higher genus, it describes a sequence of higher derivative F-term interactions of the form W^{2g}, where W is the chiral superfield of $N = 2$ supergravitational multiplet [4]. The relation between topological field theory and four-dimensional F-terms has its origin in the topological twist which projects on the chiral sector of the Hilbert space, both on the world sheet and on space-time. Therefore the topological partition function F^g should be holomorphic in chiral moduli. However, there is an anomaly of the BRST current which induces a holomorphic anomaly in F^g, captured by a set of recursion relations [1]. From the space-time point of view, this non-holomorphicity is due to the propagation of massless states which leads to non-localities in the effective action.

Type II superstring theory provides a natural setting for a topological twist due to the left-right symmetry of its world sheet [5]. Although this symmetry is absent in heterotic superstring theory, there are indications that the F-terms of the corresponding $N = 1$ supersymmetric effective action are also related to topological quantities. For instance,

the one loop corrections to gauge couplings and the Kähler metric are still related to the "new" supersymmetric index [3, 6]. In this work, we study the heterotic version of the topological theory obtained by twisting the left-moving (supersymmetric) sector which has $N = 2$ superconformal symmetry. The corresponding partition function F^g is related in this case to F-terms of the type W^{2g}, where W is now the chiral $N = 1$ gauge superfield. They give rise to amplitudes involving two gauge fields and $2g - 2$ gauginos.

Unlike in the the type II case, the recursion relations describing the holomorphic anomaly of heterotic F^g couplings do not close among themselves. Instead a new class of topological quantities appears which involve correlation functions of anti-chiral operators. We will show that once these quantities are included, the recursion relations close at least in the simplest case when one considers appropriate differences between two gauge groups with no charged massless representations. From the effective field theory point of view, the new quantities correspond to F-terms of the type $\Pi^n W^{2g}$, where Π's are chiral projections of non-holomorphic functions of chiral superfields [7]. The basic F-terms of the form Π^n appear already at the tree level, giving rise in particular to interactions involving two spacetime derivatives of complex scalars and $2n{-}2$ chiral fermions. At genus g, the W^{2g} factor gives rise to additional $2g$ gauginos. All these terms are related by local supersymmetry to gravitino "mass" terms involving $2n$ fermions and $2g$ gauginos. Therefore they could play important role for supersymmetry breaking, in the presence of fermion condensates induced by non-perturbative effects.

This paper is organized as follows. In section 2, we describe the heterotic version of the topological twist and derive the corresponding partition function related to the W^{2g} F-term. In section 3, we discuss the holomorphic anomaly and the related recursion relations. They can be put in a form of a master equation which, at least at the tree-level, has a suggestive interpretation as a Ward identity of $N = 1$ space-time supersymmetry. In section 4, we present an algorithm for constructing solutions of the tree-level recursion relations

which close on Π^n terms. In section 5, we exhibit the relation of the new topological quantities appearing in the recursion relations with physical amplitudes of the heterotic superstring theory at arbitrary genus. In section 6, we give the superfield description of the corresponding interaction terms as $\Pi^n W^{2g}$ F-terms. In section 7, we work out some simple orbifold examples of the heterotic topological quantities. In section 8, we present our conclusions and discuss the difficulties encountered in generalizing the recursion relations to the case of gauge groups with massless charged matter representations. Finally, in Appendices we perform explicit field theory computations which illustrate how a tree-level Π^2 term feeds into F-terms of the type Π^3 at the tree-level (Appendix A) and ΠW^2 at the one-loop (Appendix B), with non-holomorphic couplings satisfying the recursion relations.

2. Topological partition function in the heterotic case

We begin by reviewing some of the basic features of topological field theory obtained by twisting $N = 2$ superconformal field theories that appear in string compactifications [5]. Let us first consider the left moving sector. One starts with an $N = 2$ superconformal field theory with central charge $\hat{c} = 3$ needed to describe the compactification of the ten dimensional superstring to four dimensions. The $N = 2$ algebra is generated by the energy momentum tensor T, the two supercurrents G^+, G^-, and the $U(1)$ current J. Among the states of the theory, there are special states corresponding to chiral (anti-chiral) primary fields with integral $U(1)$ charges. These states are characterized by the relation $h = q/2$ for chiral fields and $h = -q/2$ for the anti-chiral ones, h and q being their dimensions and $U(1)$ charges, respectively. Furthermore from unitarity it follows that $|q| \leq 3$. The states with $q = \pm 1$ give rise to 4-dimensional chiral (anti-chiral) supermultiplets. The identity operator (namely $q = 0$ state) gives rise to the gravitational and gauge sectors of the theory.

The topological field theory is obtained by twisting the energy momentum tensor $T \to T + \frac{1}{2}\partial J$, so that the new central charge vanishes and the conformal dimensions of the fields

$h \to h - q/2$. In particular G^+ acquires dimension one and G^- dimension two. Moreover, from the $N = 2$ algebra, it is consistent to identify G^+ with the BRST current of the topological theory $Q_{BRST} = \oint G^+$. The modified stress energy tensor becomes BRST-exact:

$$T = \{Q_{BRST}, G^-\} \tag{2.1}$$

as it should be for a topological theory. The $U(1)$ current is now anomalous, giving rise to a background charge equal to $3(g-1)$ on a genus g Riemann surface. The chiral primaries have dimension zero and are the physical states, as defined by the BRST-cohomology. The anti-chiral ones become unphysical as they are not annihilated by Q_{BRST}. Equation (2.1) suggests that one can identify G^- with the usual reparametrization ghost b.

In the left-right symmetric theories discussed in [1], one can also twist the right moving $N = 2$ algebra $\bar{T} \to \bar{T} + \frac{1}{2}\bar{\partial}\bar{J}$.[1] G^-'s of left and right sectors can then be folded with holomorphic and anti-holomorphic Beltrami differentials to define a measure over the moduli-space of Riemann surfaces. Since the complex dimension of the moduli space M_g of genus g Riemann surface is $3(g-1)$, the charges of these G^-'s cancel exactly the background charge mentioned above. As a result, after integrating over M_g, we get the partition function F^g of a topological string theory.

One can also define correlation functions of n (chiral, chiral) operators of charge (1,1) each, by inserting them at different points of the Riemann surface. Since every insertion introduces a puncture in the surface, the dimension of the moduli space increases by n and therefore the measure involves n additional G^-'s and once again the background charge is balanced. Corresponding to these zero-form operators, one has as usual two-form operators, carrying charge zero, which can be added to the action, representing the holomorphic moduli of the target space. In our case they are the (1,2) (complex structure) moduli. On the other hand the anti-holomorphic (1,2) moduli are BRST-exact in the topological theory,

[1] For concreteness we are considering here type B-models with the same twist in the left- and right-moving sectors.

and therefore one expects that the topological correlation functions would be holomorphic. However this naive expectation turns out to be wrong, due to the presence of boundary terms in the integration over M_g [1], as we will discuss later. (1,1) (Kähler) moduli (and their complex conjugates) are also BRST-exact and can be shown to be truly decoupled.

In Refs. [1, 4] it was shown that certain physical amplitudes in type II string theory corresponding to F-terms in low energy $N=2$ supergravity theory, are given by F^g. They involve 2 gravitons and $2(g-1)$ graviphotons. To explain why these amplitudes are related to F^g, let us bosonize the left $U(1)$ current $J=i\sqrt{3}\partial H$. Then twisting of the stress energy tensor can be accomplished by adding the term $i\frac{\sqrt{3}}{2}HR^{(2)}$ to the sigma model action, where $R^{(2)}$ is the 2-dimensional world sheet curvature. Now on a genus g surface, given the fact that the Euler number is $2(1-g)$, one can choose a metric so that $R^{(2)}=-\sum_{i=1}^{2g-2}\delta^{(2)}(x-x_i)$. This therefore suggests that the topological partition function can be viewed as computing an amplitude in the untwisted theory involving $2(g-1)$ vertices whose left moving part is of the form $e^{-i\frac{\sqrt{3}}{2}H}$. Similarly bosonizing the right moving $U(1)$ current and twisting one finds that these $2(g-1)$ vertices are exactly the internal part of the graviphoton vertex operators in the $-1/2$ ghost picture. Conservation of superghost charge then implies an insertion of $3(g-1)$ picture changing operators from both left and right sectors. By $U(1)$ charge conservation the only non vanishing terms in the picture changing operators involve G^-'s giving rise to the right number of G^- insertions as is required in the definition of the topological partition function.

To realize precisely what amplitude has to be computed, we recall that the anti-selfdual part of the graviphoton field strength $T_{\mu\nu}$ is the lowest component of a chiral $N=2$ superfield $W^{ij}_{\mu\nu}$, from which the scalar superfield W^2 can be constructed [8]:

$$W^2\equiv\epsilon_{ij}\epsilon_{kl}W^{ij}_{\mu\nu}W^{kl}_{\mu\nu}=T_{\mu\nu}T_{\mu\nu}-2(\epsilon_{ij}\theta^i\sigma_{\mu\nu}\theta^j)R_{\mu\nu\lambda\rho}T_{\lambda\rho}-(\theta^i)^2(\theta^j)^2R_{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}+\ldots \tag{2.2}$$

Here $R_{\mu\nu\lambda\rho}$ is the anti-selfdual part of the Riemann tensor. It is then clear that an F-term of the type W^{2g} will contribute to an amplitude involving $2g-2$ graviphotons and 2 gravitons.

One is thus led to compute the following amplitude:

$$A^g \;=\; \int_{M_g} \langle \int \prod_{i=1}^{2g} d^2x_i V_h(x_1) V_h(x_2) \prod_{i=3}^{2g} V_T(x_i) \prod_{k=1}^{3g-3} |(\mu_k b)|^2 e^{\varphi} G^-(z_k) e^{\widetilde{\varphi}} \widetilde{G}^-(\bar{z}'_k) \rangle_g \,. \qquad (2.3)$$

Here V_T and V_h are the vertex operators for the graviphoton and graviton in $-1/2$ and 0 pictures respectively. b is the reparametrization ghost, μ_i's are the Beltrami differentials, φ is the scalar bosonizing the superghost system and the tilde refers to the right moving sector.

It was shown [4] that, after using bosonization formulae and the Riemann theta identity for summing over spin structures, all non-zero mode determinants of ghosts, spacetime bosons and fermions cancel, and the amplitude becomes proportional to:

$$A^g \;=\; \int_{M_g} \frac{1}{(\det \mathrm{Im}\tau)^2} \int \prod_{i=1}^{2g} d^2x_i |\det \omega_i(y_j) \det \omega_i(u_j)|^2 \langle \prod_{k=1}^{3g-3} |(\mu_k G^-)|^2 \rangle_{\mathrm{top}}, \qquad (2.4)$$

where g y's and g u's are partitions of the $2g$ x's, the precise partition depending on the choice of kinematics. ω_i's are the g holomorphic abelian differentials. $(\det \mathrm{Im}\tau)^{-2}$ arises from the integration over spacetime loop momenta, τ being the period matrix of the Riemann surface. The subscript top means that the correlator is evaluated in the internal twisted theory. The integration over x's cancels then $(\det \mathrm{Im}\tau)^{-2}$ and as a result we end up with just the topological partition function.

Under Kähler transformations $K \to K + \phi + \bar{\phi}$, ϕ being a holomorphic function, A^g transforms as $A^g \to e^{(g-1)(\phi-\bar{\phi})} A^g$. This follows from the Kähler transformation properties of the graviphoton in $N=2$ supergravity [9], $T_{\mu\nu} \to e^{-\frac{1}{2}(\phi-\bar{\phi})} T_{\mu\nu}$. Recalling that the F^g transforms as $F^g \to e^{(2g-2)\phi} F^g$, i.e. it has Kähler weight $(2g-2, 0)$,[2] we see that the A^g and F^g are related by $A^g = e^{(1-g)K} F^g$.

After this discussion of the type II case, it is not difficult to guess which amplitudes in the heterotic case are related to the heterotic version of the topological theory. Let us recall that in the four dimensional heterotic string with $N=1$ supersymmetry the internal

[2] In general, for a field Φ transforming as $\Phi \to e^{(w\phi + \bar{w}\bar{\phi})} \Phi$ the Kähler weight is defined to be $(w, \bar{w})$.

theory has an $N=2$ superconformal symmetry with $c=9$ in the left-moving sector, while the right-moving sector has in general just conformal symmetry with $\bar{c}=22$. Therefore one expects to have twisting only in the left-moving sector. From the previous discussion it is then clear that we are led to consider amplitudes involving $2g-2$ left-moving spin fields at genus g. Vertex operators having this form are those corresponding to gauginos, gravitinos or dilatinos. Let us first consider gauginos. An F-term giving rise to $2g-2$ gauginos can be written as $(W^2)^g$, where W^a_α is now the chiral gauge superfield:

$$W^a_\alpha = -i\lambda^a_\alpha - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha^\beta F^a_{\mu\nu}\theta_\beta + \ldots \tag{2.5}$$

Here a is the gauge group index and λ is the gaugino field. Such an F-term would contribute to an amplitude involving $2g-2$ gauginos and 2 gauge fields.

The vertex operator for gauginos, in the $-1/2$ picture is:

$$V^\alpha_\lambda(p) \;=\; :e^{-\frac{1}{2}\varphi} S^\alpha e^{i\frac{\sqrt{3}}{2}H} \bar{J}^a e^{ip\cdot X}: \;, \tag{2.6}$$

where S^α is spacetime spin field and $\bar{J}^a$ is the right-moving Kac-Moody current. The vertex operator for the gauge fields, in the 0-ghost picture, is:

$$V_A(p) \;=\; :(\partial X_\mu + ip\cdot\psi\psi_\mu)\bar{J}^a e^{ip\cdot X}: \;, \tag{2.7}$$

where ψ's are spacetime fermions. The amplitude we are interested in is therefore:

$$A^g \;=\; \int_{M_g} \langle \int \prod_{i=1}^{2g} d^2x_i V_A(x_1)V_A(x_2) \prod_{i=3}^{2g} V^\alpha_\lambda(x_i) \prod_{k=1}^{3g-3} |(\mu_k b)|^2 e^\varphi G^-(z_k)\rangle_g \;. \tag{2.8}$$

Note that the left-moving part of the gauginos and gauge fields vertices is identical to that of graviphotons and gravitons of the type II case. Therefore, following the same steps as before, and summing over left-moving spin structures, once again we find that the non-zero mode part of the left-moving ghost and spacetime boson-fermion systems cancel, and one ends up with the following expression, up to second order in the external momenta:

$$A^g \;=\; \int_{M_g} \frac{1}{(\det \mathrm{Im}\tau)^2} \int \prod_{i=1}^{2g} d^2x_i \det\omega_i(y_j)\det\omega_i(u_j)$$

$$\langle \prod_{k=1}^{3g-3} (\mu_k G^-)(\bar{\mu}_k \bar{b}) \prod_{l=1}^{g} \bar{J}^{b_l}(\bar{y}_l) \bar{J}^{c_l}(\bar{u}_l) \rangle_{\rm top} \ , \qquad (2.9)$$

where, as in the type II case, u's and y's are partitions of x's while b's and c's are the gauge group indices of the currents at positions y and u, respectively. The subscript top means that the correlator is evaluated in the "heterotic topological theory" (HTT) , which consists of the left moving twisted internal theory together with the full right-moving $\bar{c} = 26$ bosonic string. Notice however that, through the factor $(\det {\rm Im}\tau)^{-2}$, we have explicitly extracted the zero mode contribution of the spacetime bosons X_μ's. Therefore, given the fact that the momenta of the non-compact bosons are left-right symmetric, the above definition of the topological theory is not strictly rigorous. A correct definition of heterotic topological theory will involve also the twisting of the left-moving spacetime SCFT. However for the purposes of this paper, we shall not be needing this more rigorous formulation.

The physical states of the HTT are left-moving chiral operators, corresponding to spacetime chiral superfields. The two form versions of anti-chiral operators are BRST-exact and therefore one would expect them to decouple. However, insertion of such operators in the correlator (2.9) does not give a total derivative on M_g due to the presence of an explicit moduli dependent prefactor (the $(\det {\rm Im}\tau)^{-2}$ term and the x_i integrals). Thus, to proceed further, we have to perform the x_i integrals.

Let us first choose the currents along the Cartan generators (this can always be done for small enough g) so that there is no first order pole in the OPE of any two currents. The double poles can be canceled by taking suitable differences between two different gauge groups. The only contributions then come from the zero modes of the Kac-Moody currents, which are given by $\sum_{i=1}^g Q_i^a \bar{\omega}_i$, where Q_i^a measures the a-th charge of the state propagating around the i-th loop. The x_i integrations can then be done explicitly providing a factor $(\det {\rm Im}\tau)^2$, and the final result is:

$$A^g \ = \ \int_{M_g} \langle \prod_{k=1}^{3g-3} (\mu_k G^-)(\bar{\mu}_k \bar{b}) (\det Q_i^{b_j})(\det Q_i^{c_j}) \rangle_{\rm top}. \qquad (2.10)$$

The above expression, which is not modular invariant, is actually just one of the terms of the modular invariant, linear combination involving the two gauge groups which ensures the cancellation of all double poles.

The Kähler transformations of A^g can be deduced as before from the Kähler transformations of gauginos $\lambda \to e^{-\frac{1}{4}(\phi-\bar{\phi})}\lambda$. Thus A^g transforms as $A^g \to e^{\frac{g-1}{2}(\phi-\bar{\phi})}A^g$. Here again one can define $F^g = e^{\frac{(g-1)}{2}K}A^g$ which transforms holomorphically $F^g \to e^{(g-1)\phi}F^g$, carrying Kähler weight $(g-1,0)$. When discussing the recursion relations in the following section, we will adopt, both for type II and heterotic, the prescription of Ref. [1] for the normalization of the vacuum state in the topological theory, so that the topological partition function computes directly F^g's.

3. Recursion relations

In this section we will discuss the holomorphic anomaly and the related recursion relations for the heterotic string. Before doing that, it is useful to recall type II case. Taking the antiholomorphic derivative of the topological partition function F^g amounts to the insertion of the two form version $\Phi_{\bar{\imath}}$ of the corresponding (anti-chiral,anti-chiral) operator $\Psi_{\bar{\imath}}$. As we have already mentioned in the previous section, $\Phi_{\bar{\imath}} = \{Q_{BRST}, \{\bar{Q}_{BRST}\Psi_{\bar{\imath}}\}\}$. Starting from the one point function of $\Phi_{\bar{\imath}}$ and deforming the BRST contours one gets (for $g \geq 2$):

$$\partial_{\bar{\imath}}F_g \;=\; \sum_{a,b=1}^{3g-3}\int_{M_g}(-)^{(a+b)}\langle\prod_{c\neq a,d\neq b}(\mu_c G^-)(\bar{\mu}_d\bar{G}^-)(\mu_a T)(\bar{\mu}_b\bar{T})\int d^2z\Psi_{\bar{\imath}}(z,\bar{z})\rangle\;. \tag{3.1}$$

The insertions of the energy momentum tensors T and $\bar{T}$ give rise to total derivatives in the moduli space M_g. It follows that possible contributions to $\partial_{\bar{\imath}}F^g$ arise from the boundaries of M_g, i.e. from Riemann surfaces with nodes.

The analysis of the boundary contributions is very similar in the two cases of pinching a handle or a dividing geodesic: parameterize the opening of a node by the complex coor-

dinate t, then due to the fact that in (3.1) one has $\partial_t \partial_{\bar{t}}$ acting on the one point function of $\Psi_{\bar{\imath}}$ on a surface with node, the only non vanishing contribution comes when this one point function behaves like $\ln(t\bar{t})$ as $t \to 0$. This behavior appears when the operator $\Psi_{\bar{\imath}}$ sits in the node and interacts with two anti-chiral operators via Yukawa coupling $C_{\bar{\imath}\bar{\jmath}\bar{k}}$, the integration of its position giving $\ln(t\bar{t})$. Finally the two anti-chiral operators propagate to chiral ones giving rise to holomorphic derivatives of lower genus partition function. Thus one gets the following recursion relation:

$$\partial_{\bar{\imath}} F^g = \frac{1}{2} C_{\bar{\imath}\bar{\jmath}\bar{k}} e^{2K} G^{i\bar{\imath}} G^{j\bar{\jmath}} [\ \sum_{g'=1}^{g-1} D_j F^{g'} D_k F^{g-g'} + D_j D_k F^{g-1}\ ] \tag{3.2}$$

Here K is the Kähler potential and $G^{i\bar{\imath}}$ is the inverse of the Kähler metric and D is the Kähler covariant derivative. Since F^g transforms as $F^g \to e^{(2g-2)\phi} F^g$ under the Kähler transformations, we have $D_i F^g \equiv (\partial_i + (2g-2)K_i) F^g$.[3] The crucial point here is that due to the presence of $\partial_t \partial_{\bar{t}}$, the extra operator $\Psi_{\bar{\imath}}$ gets stuck in the node and as a result what appears in the right hand side of (3.2) again involves holomorphic derivatives of partition functions. In other words recursion relations close within the set of all partition functions. As we shall see below in heterotic case this does not happen.

Indeed let us take a derivative of F^g (related to the A^g of (2.10) as explained in the previous section) with respect to some anti-holomorphic modulus. The two form $\Phi_{\bar{\imath}}$ corresponding to this modulus is again BRST-exact, $\Phi_{\bar{\imath}} = \{Q_{BRST}, \Psi_{\bar{\imath}}\}$, where $\Psi_{\bar{\imath}}$ is dimension $(1,1)$ anti-chiral operator with charge (-1). Note that now only one BRST (left-moving) operator appears in this expression. Inserting $\Phi_{\bar{\imath}}$ in F^g and deforming the BRST-contour one finds:

$$\partial_{\bar{\imath}} F^g \ = \ \int_{M_g} \sum_{m=1}^{3g-3} (-1)^m \langle (\mu_m T) \prod_{n \neq m} (\mu_n G^-) \prod_{k=1}^{3g-3} (\bar{\mu}_k \bar{b}) \int \Psi_{\bar{\imath}} (\det Q_i^{b_j})(\det Q_i^{c_j}) \rangle_{\rm top}. \tag{3.3}$$

Once again we get a total derivative in the world-sheet moduli space and the contribution coming from the boundaries corresponding to Riemann surfaces with nodes is ∂_t of the

[3] In general, on a quantity of Kähler weight $(w, \bar{w})$ the Kähler covariant derivatives are $(\partial_i + wK_i)$ and $(\partial_{\bar{\imath}} + \bar{w}K_{\bar{\imath}})$, respectively.

one point function of $\Psi_{\bar\imath}$. The integral over $t, \bar t$ in the limit $t \to 0$ will be non-vanishing only if this one-point function behaves as $1/\bar t$. This indicates that the intermediate state going through the node is a chiral massless state (i.e. has left and right-moving dimensions equal to 0 and 1 respectively) and moreover the operator $\Psi_{\bar\imath}$ should be distributed over the complement of the node.[4] What are the possible intermediate states?

Let us first consider the case of pinching a dividing geodesic, so that the surface splits into two components Σ_1 and Σ_2 of genus g_1 and g_2 ($g_1 + g_2 = g$) with punctures P_1 and P_2, respectively. The number of G^-'s as well as $\bar b$'s on Σ_1 and Σ_2 is $(3g_1 - 3 + 1)$ and $(3g_2 - 3 + 1)$, respectively, due to the fact that there is one extra world-sheet modulus for each puncture. We can assume that the operator $\Psi_{\bar\imath}$ is on Σ_1. Since the operator $\Psi_{\bar\imath}$ carries $U(1)$ charge (-1), the anomalous conservation of the $U(1)$ charge implies that the intermediate state at the puncture P_1 must have a charge $(+2)$. This further implies that the state at P_2 being dual to the one at P_1 must have a charge $(+1)$ which is also consistent with the $U(1)$ anomaly on Σ_2. The right moving part of the intermediate state must be gauge group singlet. On Σ_2 using the extra G^-, we can convert the charge $(+1)$ state into neutral two-form operator which just corresponds to taking a holomorphic Kähler covariant derivative D_j of F^{g_2}. Note that here j labels any chiral gauge singlet and not necessarily only moduli. On Σ_1 the charge $(+2)$ state of dimension $(0,0)$ appearing at the puncture can be identified with the state obtained by the action of the holomorphic three form ρ on an anti-chiral charge (-1) state $\bar c \Psi_{\bar\jmath}$: $\oint dz \rho(z) \bar c \Psi_{\bar\jmath}$. Here $\bar c$ is the right-moving reparametrization ghost.[5] Thus on Σ_1 we end up with a new object $F^{g_1}_{\bar\imath;\bar\jmath}$ that is a two point function involving an unphysical charge (-1) operator $\Psi_{\bar\imath}$ and the charge (+2) operator $\oint \rho \Psi_{\bar\jmath}$. Therefore the

[4]In fact if there is a non-zero contribution when $\Psi_{\bar\imath}$ is at the node then this would result in a logarithmic divergence due to the appearance of $\ln(t\bar t)$. This could occur if there are massless states in the theory which acquire a mass in the presence of non-zero vacuum expectation value for the $\bar\imath$- modulus. In the following we assume that we are dealing with a region of moduli space where massless states remain massless.

[5]Recall that ρ is a chiral charge $(+3)$ operator with dimension 0 and exists for all internal theories leading to space-time supersymmetry.

holomorphic anomaly is given by:

$$\partial_{\bar{i}} F^g \;=\; \sum_{g_1+g_2=g} F^{g_1}_{\bar{i};\bar{j}} G^{\bar{j}j} D_j F^{g_2} + \cdots \tag{3.4}$$

where $g_1, g_2 > 0$ and the dots refer to contributions coming from pinching a handle. Notice that in order for this equation to make sense, $F^{g_1}_{\bar{i};\bar{j}}$ must carry Kähler weight $(g_1, 0)$, which indeed turns out to be the case as we shall see below.

In the case of pinching a handle, we expect a contribution from a configuration in which a $(Q^a)^2$ is inserted on the handle itself. Thus, one obtains a two-point function on genus $g-1$ Riemann surface involving a massless state charged under the gauge group. As in the dividing case, after the pinching there are $3g-4$ G^-'s, two of which are used to convert the states at the punctures in their integrated two-form version, while the remaining $3g-6$ are used to define the genus $g-1$ two-point function. To simplify the discussion, we will consider models with pure gauge groups without charged matter fields. In this case the handle contribution vanishes for the following reason: the two conjugate states propagating in the handle involve the identity operator and its conjugate three-form ρ in the internal $N=2$ superconformal theory, therefore acting with G^- on the identity we get zero. In section 7, we discuss explicit construction of such models, while in the last section we comment on difficulties concerning the generalization to the case with charged matter fields.

Unlike the type II case, the appearance of new objects besides F^g's on the right hand side of equation (3.4) shows that the recursion relations do not close within the topological correlation functions. Indeed from the definition of $F^{g_1}_{\bar{i};\bar{j}}$ it is clear that we have to allow operators in the twisted theory that are not in the kernel of Q_{BRST}. Nevertheless we shall see below that these quantities are also related to physical amplitudes in heterotic string theory. One can ask the question whether one can obtain some recursion relations for $F^{g_1}_{\bar{i};\bar{j}}$. To see this let us take an anti-holomorphic derivative of $F^{g_1}_{\bar{i};\bar{j}}$. This amounts to insertion of $\Phi_{\bar{k}} = \{Q_{BRST}, \Psi_{\bar{k}}\}$ in this two point function. This time however the deformation of BRST-

contour does not give only boundary terms due to the fact that there is a contribution when Q_{BRST} acts on $\Psi_{\bar{\imath}}$. Note that Q_{BRST} annihilates $\oint \rho\Psi_{\bar{\jmath}}$ as it is expected from the fact that the latter is a charge $(+2)$ chiral operator. If one anti-symmetrizes $\bar{k}$ and $\bar{\imath}$ then only the boundary terms survive, i.e. those coming from the action of Q_{BRST} on the G^-'s. One can then repeat the analysis which led to (3.4) and the result from pinching a dividing geodesic is:

$$D_{[\bar{k}}F^g_{\bar{\imath}];\bar{\jmath}} = \sum_{g_1+g_2=g} G^{m\bar{m}}(F^{g_1}_{\bar{k}\bar{\imath};\bar{\jmath}\bar{m}}D_m F^{g_2} + F^{g_1}_{[\bar{k};\bar{m}}D_m F^{g_2}_{\bar{\imath}];\bar{\jmath}}) \ , \tag{3.5}$$

where square brackets denote anti-symmetrization of $\bar{k}$ and $\bar{\imath}$. Notice that on the right hand side of (3.5) there appears yet another new object with four indices $F^{g_1}_{\bar{k}\bar{\imath};\bar{\jmath}\bar{m}}$ which is defined as the 4-point function involving $\Psi_{\bar{k}}$, $\Psi_{\bar{\imath}}$, $\oint dz\rho(z)\bar{c}\Psi_{\bar{\jmath}}$ and $\oint dz\rho(z)\bar{c}\Psi_{\bar{m}}$. Since Ψ's are anti-commuting this 4-point function is anti-symmetric in $\bar{k}$ and $\bar{\imath}$, and similarly due to the presence of $\bar{c}$ it is also anti-symmetric in $\bar{\jmath}$ and $\bar{m}$.

It is clear now that repeating this procedure one would keep getting higher and higher point functions. The general structure of these terms is

$$\begin{aligned} F^g_{\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1\cdots\bar{\jmath}_n} \ &= \ \int_{M_{g,n}} \langle \prod_{k=1}^{3g-3+n} (\mu_k G^-)(\bar{\mu}_k\bar{b})(\det Q_i^{b_j})(\det Q_i^{c_j}) \\ &\qquad \int \Psi_{\bar{\imath}_1}\cdots\int \Psi_{\bar{\imath}_n}\widetilde{\Psi}_{\bar{\jmath}_1}\cdots\widetilde{\Psi}_{\bar{\jmath}_n}\rangle_{\rm top}. \end{aligned} \tag{3.6}$$

where $\widetilde{\Psi}_{\bar{\jmath}} = \oint dz\rho(z)\bar{c}\Psi_{\bar{\jmath}}$. Here $M_{g,n}$ is the moduli space of genus g Riemann surface with n punctures corresponding to the insertion of $\widetilde{\Psi}_{\bar{\jmath}}$'s. Note that $\Psi_{\bar{\imath}}$'s being of dimension $(1,1)$ are integrated over the surface. As in the $n=2$ case, $F^g_{\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1\cdots\bar{\jmath}_n}$ is totally antisymmetric in the $\bar{\imath}$ and $\bar{\jmath}$ indices. Moreover one can prove the following identity:

$$F^g_{[\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1]\bar{\jmath}_2\cdots\bar{\jmath}_n} \ = \ 0. \tag{3.7}$$

Here the square bracket indicates total antisymmetrization of all the enclosed indices $[\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1]$. To see this, convert the $\widetilde{\Psi}_{\bar{\jmath}_1}$ operator into its two form version by acting with G^- and $\bar{b}$:

$$\oint dzG^-(z)\oint d\bar{z}\bar{b}(\bar{z})\widetilde{\Psi}_{\bar{\jmath}} = \oint dz\mathcal{J}(z)\Psi_{\bar{\jmath}_1}, \tag{3.8}$$

where $\mathcal{J} = \oint dz G^-(z)\rho$ is a charge (+2) bosonic current, which in the twisted theory has dimension 1. We can now substitute eq.(3.8) into (3.6) and deform the contour of $\mathcal{J}$. Using the fact that it annihilates $\widetilde{\Psi}$ and also the antisymmetry properties of Ψ's one immediately gets the identity (3.7).

Let us now turn to the question whether these higher point functions in the twisted theory have a counterpart in the heterotic string theory. As we have already seen the partition function corresponds to a string amplitude involving 2 gauge fields and $(2g-2)$ gauginos. By the internal $U(1)$ conservation as well as superghost charge conservation this amplitude involves $3g-3$ G^-'s as is required for topological partition function. For the $2n$-point function above we have $3g-3+n$ G^-'s. First of all since these insertions correspond to anti-chiral operators we must consider a string amplitude involving $2n$ additional anti-chiral fields. We will insert $2n$ anti-chiral fermions $\bar{\chi}_{\bar{\imath}}$ whose vertex operators in the $-1/2$ ghost picture are given by:

$$V_{\bar{\imath}}^{\dot{\alpha}} = e^{-\frac{\varphi}{2}} S^{\dot{\alpha}} \Sigma_{\bar{\imath}} e^{ip\cdot X}, \tag{3.9}$$

where $\Sigma_{\bar{\imath}} = \oint dz \frac{1}{\sqrt{z}} e^{i\frac{\sqrt{3}}{2}H(z)} \Psi_{\bar{\imath}}$ carries $U(1)$ charge $(+1/2)$. This necessitates the presence of n additional picture changing operators. From $U(1)$ charge conservation it follows that only the part of the picture changing operators containing G^- contributes to the amplitude.

We are thus led to consider a genus g string amplitude of the form $\langle F_{\mu\nu}^2 \lambda^{2g-2} \bar{\chi}^{2n} \rangle$. In section 5, we show that indeed, after using bosonization formulae and summing over spin structures, this amplitude is proportional to $e^{\frac{1}{2}(g-1+n)K} F^g_{\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1\cdots\bar{\jmath}_n}$, the two types of indices $\bar{\imath}$ and $\bar{\jmath}$ corresponding to the two spin fields $S^{\dot{1}}$ and and $S^{\dot{2}}$. We shall also see in section 6 that these amplitudes are related to some higher weight F-terms in the supergravity action. From this identification it also follows that the Kähler weight of $F^g_{\bar{\imath}_1\cdots\bar{\imath}_n\ ;\ \bar{\jmath}_1\cdots\bar{\jmath}_n}$ is $(g-1+n, 0)$, since the $\bar{\chi}$'s transform as λ's.

The arguments leading to (3.5) can now be used to obtain the recursion relation for

$F^g_{\bar\imath_1\cdots\bar\imath_n\ ;\ \bar\jmath_1\cdots\bar\jmath_n}$, with the result

$$D_{[\bar\imath}F^g_{\bar\imath_1\cdots\bar\imath_n]\ ;\ \bar\jmath_1\cdots\bar\jmath_n} \quad = \quad \sum_{g_1+g_2=g}\sum_{m=0}^{n}\sum_{\text{partitions}} F^{g_1}_{\bar k_1\cdots\bar k_{m+1}\ ;\ \bar\ell_1\cdots\bar\ell_m\bar\ell}G^{\ell\bar\ell}D_\ell F^{g_2}_{\bar k_{m+2}\cdots\bar k_{n+1}\ ;\ \bar\ell_{m+1}\cdots\bar\ell_n}\ . \tag{3.10}$$

Here $\bar k_1\cdots\bar k_{n+1}$ are different permutations of $\bar\imath,\bar\imath_1\cdots\bar\imath_n$ while $\bar\ell_1\cdots\bar\ell_n$ are permutations of $\bar\jmath_1\cdots\bar\jmath_n$. The sum over permutations is weighted by their respective signs. For $g_1=0$ the range of m is between 1 and n while for $g_2=0$ the range is between 0 and $(n-2)$. Notice that this equation is consistent with the Kähler weight assignments we have given above.

If one considers $F^g_{\bar\imath_1\cdots\bar\imath_n\ ;\ \bar\jmath_1\cdots\bar\jmath_n}$ as a differential form in the $\bar\imath$ indices, then eq.(3.10) is a statement about its non-closedness. This equation can be further put in a succinct form in terms of a generating function. Recalling that (3.6) is totally anti-symmetric in $\bar\imath$ and $\bar\jmath$ indices separately, we introduce Grassman parameters $\theta^{\bar\imath}$ and $\eta^{\bar\jmath}$. One can now define the generating function as:

$$F^g = -\sum_{n\geq 2}\frac{1}{(n!)^2}\theta^{\bar\imath_1}\eta^{\bar\jmath_1}\cdots\theta^{\bar\imath_n}\eta^{\bar\jmath_n}F^g_{\bar\imath_1\cdots\bar\imath_n\ ;\ \bar\jmath_1\cdots\bar\jmath_n} \tag{3.11}$$

The identity (3.7) in terms of the generating function F^g becomes:

$$\theta^{\bar\imath}\frac{\partial F^g}{\partial\eta^{\bar\imath}} = 0. \tag{3.12}$$

Finally the recursion relation (3.10) can then be recast as:

$$\theta^{\bar\imath}D_{\bar\imath}F^g = \sum_{g_1+g_2=g}\frac{\partial F^{g_1}}{\partial\eta^{\bar\jmath}}G^{j\bar\jmath}D_jF^{g_2}. \tag{3.13}$$

Here it is understood that the covariant derivatives act just on the coefficients $F^g_{\bar\imath_1\cdots\bar\imath_n\ ;\ \bar\jmath_1\cdots\bar\jmath_n}$, and not on θ's and η's. In other words $D_{\bar\imath}F^g = \partial_{\bar\imath}F^g - \Gamma^{\bar k}_{\bar\imath\bar\jmath}(\theta^{\bar\jmath}\frac{\partial}{\partial\theta^k} + \eta^{\bar\jmath}\frac{\partial}{\partial\eta^k})F^g$ and $D_iF^g = \partial_iF^g + K_i(\eta^{\bar\jmath}\frac{\partial}{\partial\eta^{\bar\jmath}} - 1)F^g$.

An important consistency condition for (3.13) is provided by the integrability condition $(\theta^{\bar\imath}D_{\bar\imath})^2F^g = 0$. Indeed one can verify by using antisymmetry properties of Grassman variables, equation (3.13) and the identity (3.12) that the operator $\theta^{\bar\imath}D_{\bar\imath}$ annihilates the right hand side of (3.13).

Specializing eq.(3.10) to the case $g = 0$ (where there is obviously no handle degeneration in any model), the anomaly equation becomes:

$$\theta^{\bar{\imath}} D_{\bar{\imath}} F^0 = \frac{\partial F^0}{\partial \eta^{\bar{\jmath}}} G^{j\bar{\jmath}} D_j F^0. \tag{3.14}$$

Equation (3.14) is strongly suggestive of space-time supersymmetry Ward identity. Consider the anti-chiral super multiplet $(z^{\bar{\imath}}, \bar{\chi}^{\bar{\imath}}, h^{\bar{\imath}})$ where $h^{\bar{\imath}}$ is the auxiliary field and $\bar{\chi}^{\bar{\imath}}$ is a two component spinor with the two components being $\theta^{\bar{\imath}}$ and $\eta^{\bar{\imath}}$. One of the supersymmetry transformations on these fields is:

$$\begin{aligned} \delta_s z^{\bar{\imath}} &= \theta^{\bar{\imath}} \\ \delta_s \eta^{\bar{\imath}} &= -h^{\bar{\imath}} \\ \delta_s \theta^{\bar{\imath}} &= \delta_s h^{\bar{\imath}} = 0, \end{aligned} \tag{3.15}$$

and the variation of all holomorphic fields is zero. Then the Ward identity for such a transformation on the effective action S is:

$$\theta^{\bar{\imath}} \partial_{\bar{\imath}} S = -h^{\bar{\imath}} \frac{\partial}{\partial \eta^{\bar{\imath}}} S \tag{3.16}$$

If now we identify S with F^0 and $-h^{\bar{\imath}}$ with $(\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \theta^{\bar{\jmath}} \eta^{\bar{k}} + G^{i\bar{\imath}} D_i F^0)$ then we obtain (3.14). The above argument was heuristic, however in the following section where we describe a method to construct a solution of the recursion relations, we shall make this argument more precise. It is also interesting to note that a similar result has been obtained recently [10] for the $N = 2$ recursion relation (2). The first term on the r.h.s. which originates from handle degeneration has been found as a consequence of $N = 2$ space-time supersymmetry.

4. Solution to the tree level recursion relations

We now present a topological Feynman diagrammatic solution to the recursion relation (3.14). The first point to note is that there is an ambiguity in the solution. Since eq.(3.14)

(or (3.10) with $g = 0$) relates n-forms[6] on the left hand side to lower forms on the right hand side, at each step one can always add a closed n-form to a given solution and get another solution. This is analogous to the holomorphic ambiguity in the type II case. In fact these closed n-forms are associated to the true F-terms in the effective action, to be discussed in section 6. They represent contributions of massive string modes as contrasted to the non-closed part which is generated by connected graphs with vertices corresponding to lower forms and massless propagators. Our solution will be obtained by starting with a set of closed n-forms ($n \geq 2$), which can be thought of as genuine F-terms from the effective field theory point of view. Thus consider the n-form

$$F^{[n]} = F^0_{\bar{\imath}_1,\ldots,\bar{\imath}_n;\bar{\jmath}_1,\ldots,\bar{\jmath}_n} \theta^{\bar{\imath}_1} \cdots \theta^{\bar{\imath}_n} \eta^{\bar{\jmath}_1} \cdots \eta^{\bar{\jmath}_n} \tag{4.1}$$

which satisfies:

$$\delta F^{[n]} \equiv \theta^{\bar{\imath}} \partial_{\bar{\imath}} F^{[n]} = 0, \qquad\qquad \theta^{\bar{\imath}} \frac{\partial F^{[n]}}{\partial \eta^{\bar{\imath}}} = 0 \tag{4.2}$$

Under Kähler transformation we assign weight $(-\frac{1}{2}, 0)$ to θ's and η's, so that $F^{[n]}$ have weight $(-1, 0)$.

We would now like to construct the vertices corresponding to $F^{[n]}$. The set of vertices can be grouped together in the following n-th order polynomial in the $M+1$ variables $h^{\bar{A}}$, $A = 0$ or i with $i = 1, \ldots, M$ where M is the number of matter fields:

$$\tilde{F}^{[n]} = F^{[n]} + \sum_{k=1}^{n} F^{(n,k)}_{\bar{A}_1 \ldots \bar{A}_k} h^{\bar{A}_1} \ldots h^{\bar{A}_k} \tag{4.3}$$

with the coefficients $F^{(n,k)}$ being functions of $z^i, z^{\bar{\imath}}, \theta^{\bar{\imath}}, \eta^{\bar{\imath}}$. We require that $\tilde{F}^{[n]}$ satisfy the equations:

$$\begin{aligned} \delta \tilde{F}^{[n]} &= h^{\bar{\imath}} \frac{\partial \tilde{F}^{[n]}}{\partial \eta^{\bar{\imath}}} + \theta^{\bar{\imath}} h^{\bar{0}} \frac{\partial \tilde{F}^{[n]}}{\partial h^{\bar{\imath}}}, \\ \theta^{\bar{\imath}} \frac{\partial \tilde{F}^{[n]}}{\partial \eta^{\bar{\imath}}} &= 0. \end{aligned} \tag{4.4}$$

[6]The degree of the form here refers to the indices that are contracted with $\theta^{\bar{\imath}}$'s

It is clear from these equations that $F^{(n,k)}$ contain $(n-k)$ $\theta^{\bar{\imath}}$'s and $\eta^{\bar{\imath}}$'s each. Note that the above equations are in fact a statement of supersymmetry invariance of $\tilde{F}^{[n]}$'s, where we define supersymmetry transformations on the variables as:

$$\delta_s z^{\bar{\imath}} = \theta^{\bar{\imath}}, \qquad \delta_s \eta^{\bar{\imath}} = -h^{\bar{\imath}}, \qquad \delta_s h^{\bar{\imath}} = -\theta^{\bar{\imath}} h^{\bar{0}}, \qquad \delta_s z^i = \delta_s \theta^{\bar{\imath}} = \delta_s h^{\bar{0}} = 0 \tag{4.5}$$

It is clear that first of the equations (4.4) is just the statement that $\tilde{F}^{[n]}$ is invariant under this transformation. The variation δ_s is just half of the supersymmetry transformation that act on anti-chiral fields and the invariance of $\tilde{F}^{[n]}$ has the significance that it represents the lowest component of a chiral superfield. Note that its action on the fields (4.5) is not nilpotent. In section 6, we will discuss the relation of these transformations to the usual supersymmetry transformations in $N = 1$ Poincaré supergravity.

Under analytic reparametrizations: $z^i \to z'^i(z)$, we demand that $\tilde{F}^{[n]}$ be invariant provided

$$h^{\bar{\imath}} \to \frac{\partial z'^{\bar{\imath}}}{\partial z^{\bar{\jmath}}} h^{\bar{\jmath}} + \frac{\partial^2 z'^{\bar{\imath}}}{\partial z^{\bar{\jmath}} \partial z^{\bar{k}}} \theta^{\bar{\jmath}} \eta^{\bar{k}}. \tag{4.6}$$

Of course θ and η transform as vectors. Under Kähler transformations we assign weight $(-1, 0)$ to $h^{\bar{A}}$'s, as required by the consistency of equations (4.4) and (4.6) and as a result $\tilde{F}^{[n]}$ have also weight $(-1, 0)$.

For $n = 1$ we define

$$\tilde{F}^{[1]} = \theta^{\bar{\imath}} \eta^{\bar{\jmath}} K_{\bar{\imath}\bar{\jmath}} - h^{\bar{\imath}} K_{\bar{\imath}} - h^{\bar{0}} K \tag{4.7}$$

where K is the Kähler potential and subscripts on K denote partial derivatives. One can easily see that $\tilde{F}^{[1]}$ satisfies the conditions (4.4) and (4.6). The action is then defined as:

$$S = \sum_{n \geq 1} e^{(n-1)(z^0 + \phi^0)} \tilde{F}^{[n]}(z^i + \phi^i, z^{\bar{\imath}}, \theta^{\bar{\imath}}, \eta^{\bar{\imath}}, h^{\bar{A}}) - \tilde{F}^{[1]}(z^i, z^{\bar{\imath}}, \theta^{\bar{\imath}}, \eta^{\bar{\imath}}, h^{\bar{A}}) - \phi^0 h^{\bar{0}} \tag{4.8}$$

We regard ϕ^A and $h^{\bar{A}}$ as quantum variables which are to be integrated out. The variable z^0 above is just introduced for convenience and we shall set it to zero at the end. The action S is invariant under analytic reparametrization with $h^{\bar{\imath}}$ transforming as (4.6) and

ϕ^i transforming as $\phi^i \to z'^i(z+\phi) - z'^i(z)$. From the foregoing discussion it is clear that under the Kähler transformation $K \to K + f(z) + \bar{f}(\bar{z})$,

$$\tilde{F}^{[n]}(z+\phi) \to e^{(n-1)f(z+\phi)-nf(z)}(\tilde{F}^{[n]}(z+\phi) - \delta_{n,1}h^{\bar{0}}(f(z+\phi)+\bar{f}(\bar{z})) \tag{4.9}$$

The action S then transforms as weight $(-1,0)$ provided $\phi^0 \to \phi^0 - f(z+\phi) + f(z)$.

Consider the tree level effective action W which is obtained by substituting the classical solution of ϕ^A and $h^{\bar{A}}$ in the action S. The effective action W can be expanded as:

$$W = -\sum_{n\geq 1} e^{(n-1)z^0} W^{[n]}_{\bar{\imath}_1,\ldots,\bar{\imath}_n;\bar{\jmath}_1,\ldots,\bar{\jmath}_n} \theta^{\bar{\imath}_1}\eta^{\bar{\jmath}_1}\cdots\theta^{\bar{\imath}_n}\eta^{\bar{\jmath}_n} \tag{4.10}$$

where $W^{[n]}$ depend on z^i and $z^{\bar{\imath}}$'s. We have extracted above the explicit dependence on z^0 by using the scaling property of the action S under the rescalings:

$$\theta^{\bar{\imath}} \to e^{\alpha}\theta^{\bar{\imath}}, \quad \eta^{\bar{\imath}} \to e^{\alpha}\eta^{\bar{\imath}}, \quad h^{\bar{\imath}} \to e^{2\alpha}h^{\bar{\imath}}, \quad z^0 \to z^0 - 2\alpha \tag{4.11}$$

One can actually show, using the form of the action S, that $n=1$ term in the above expansion vanishes. From the transformation properties of S discussed above, it follows that the coefficients $W^{[n]}$ transform covariantly under analytic reparametrizations and have Kähler weight $(n-1,0)$

We now show that W satisfies the recursion relation (3.14) where reparametrization and Kähler covariant derivatives appear. We start by applying δ on W. Since W is the tree level effective action it follows that:

$$\delta W = (\delta + (\delta\phi^A_{\rm cl})\frac{\partial}{\partial\phi^A} + (\delta h^{\bar{A}}_{\rm cl})\frac{\partial}{\partial h^{\bar{A}}})S|_c = \delta S|_c \tag{4.12}$$

where $\phi_{\rm cl}$ denotes the classical solution and the symbol $|_c$ means that the corresponding quantity is evaluated at the classical solution. In the second step we have used the equation of motion. Using then (4.4) and once again using the equation of motion we get:

$$\delta W = h^{\bar{\imath}}_{\rm cl}\frac{\partial W}{\partial\eta^{\bar{\imath}}} \tag{4.13}$$

To find the classical solution for $h^{\bar{\imath}}$, we vary S with respect to ϕ^i and obtain

$$h^{\bar{\imath}}_{\rm cl} = G^{i\bar{\imath}}\frac{\partial W}{\partial z^i} + \theta^{\bar{\jmath}}\eta^{\bar{k}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} - G^{i\bar{\imath}}h^{\bar{0}}_{\rm cl}K_i \tag{4.14}$$

The second term on the right hand side above when substituted in (4.13) covariantizes the δ appearing on the left hand side of (4.13). Finally the classical solution for $h^{\bar{0}}$ is obtained by varying the action with respect to ϕ^0 with the result

$$h^{\bar{0}}_{\rm cl} = \frac{\partial W}{\partial z^0} = (\eta^{\bar{\imath}}\frac{\partial}{\partial \eta^{\bar{\imath}}} - 1)W \tag{4.15}$$

where in the second equality we have used the fact that W is of the form given in (4.10). This expression when substituted in (4.14) Kähler covariantizes the derivative with respect to z^i appearing on the right hand side of (4.14). Thus combining equations (4.13), (4.14) and (4.15) one finds that W satisfies the recursion relation (3.14).

Now we construct $\tilde{F}^n$ explicitly for the case when F^n is of the form described in section arising from Π^n term, namely

$$F^{[n]} = f^1_{\bar{\imath}_1\bar{\jmath}_1}\theta^{\bar{\imath}_1}\eta^{\bar{\jmath}_1}\cdots f^n_{\bar{\imath}_n\bar{\jmath}_n}\theta^{\bar{\imath}_n}\eta^{\bar{\jmath}_n} \tag{4.16}$$

Then the following $\tilde{F}^n$ satisfies the conditions (4.4):

$$\tilde{F}^{[n]} = \sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_\sigma f^{\sigma(1)}_{\bar{\imath}_1\bar{\jmath}_1}\theta^{\bar{\imath}_1}\eta^{\bar{\jmath}_1}\cdots f^{\sigma(k)}_{\bar{\imath}_k\bar{\jmath}_k}\theta^{\bar{\imath}_k}\eta^{\bar{\jmath}_k}f^{\sigma(k+1)}_{\bar{A}_1}h^{\bar{A}_1}\cdots f^{\sigma(n)}_{\bar{A}_{n-k}}h^{\bar{A}_{n-k}} \tag{4.17}$$

where $f^a_{\bar{0}} \equiv f^a$ and σ denotes an element of the permutation group of n-objects. In fact this expression for $\tilde{F}^{[n]}$ is the lowest component of the superpotential term with $h^{\bar{A}}$ being the auxiliary fields. Let us consider an example of such a term when $F^{[n]}$ is not zero only for $n = 2$. Then using (4.17) in the definition of action S and solving for the effective action we find:

$$\begin{aligned} W = F^{[2]} \quad - \quad & 2[(\nabla_{\bar{\imath}_1}\nabla_{\bar{\jmath}_1}f_\alpha)f^\alpha_{\bar{p}}G^{i\bar{p}}D_i[(\nabla_{\bar{\imath}_2}\nabla_{\bar{\jmath}_2}f_\beta)(\nabla_{\bar{\imath}_3}\nabla_{\bar{\jmath}_3}f^\beta)] \\ & +(\nabla_{\bar{\imath}_1}\nabla_{\bar{\jmath}_1}f_\alpha)f^\alpha_{\bar{p}}(\nabla_{\bar{\imath}_2}\nabla_{\bar{\jmath}_2}f_\beta)f^\beta_{\bar{q}}G^{\imath\bar{p}}G^{\ell\bar{q}}R_{\imath\bar{\imath}_3\ell\bar{\jmath}_3} \\ & +(\nabla_{\bar{\imath}_1}\nabla_{\bar{\jmath}_1}f_\alpha)f^\alpha(\nabla_{\bar{\imath}_2}\nabla_{\bar{\jmath}_2}f_\beta)(\nabla_{\bar{\imath}_3}\nabla_{\bar{\jmath}_3}f^\beta)]\theta^{\bar{\imath}_1}\eta^{\bar{\jmath}_1}\theta^{\bar{\imath}_2}\eta^{\bar{\jmath}_2}\theta^{\bar{\imath}_3}\eta^{\bar{\jmath}_3} + \cdots \end{aligned} \tag{4.18}$$

where dots refer to higher order terms and the lower index f's are $f_1 = f^2$ and $f_2 = f^1$. ∇'s are reparametrization covariant derivatives, D_i are Kähler covariant derivatives and G and R are respectively the Kähler metric and Riemann tensors. This result for W agrees with the effective field theory calculation presented in the Appendix up to a closed 3-form, which is the ambiguity in the solution of the recursion relation up to this order.

To summarize, in this section we have given a method to construct a solution to the recursion relations. Moreover we have seen that the recursion relations are a consequence of the underlying supersymmetry of the action defined in (4.5).

5. Relation with heterotic amplitudes

In section 3 we anticipated that the genus g string amplitudes

$$A^{g,n} = \langle F^2_{\mu\nu}\lambda^{2g-2}\bar{\chi}^{2n}\rangle \tag{5.1}$$

are related to the topological amplitudes $F^g_{\bar{\imath}_1\cdots\bar{\imath}_n;\bar{\jmath}_1\cdots\bar{\jmath}_n}$. In this section we are going to show this by an explicit computation in the general $(2,0)$ case.[7]

As we explained in section 2, the vertex operators for gauginos contain, in the $-1/2$ ghost picture, a combination of spin fields S_α, whose precise form is governed by kinematics. After bosonizing the space-time fermionic coordinates ψ^μ by arranging them into two complex left-moving fermions $\psi_{1,2} = e^{i\phi_{1,2}}$, we have:

$$S_1 \;=\; \exp[\frac{i}{2}(\phi_1+\phi_2)]\ , \tag{5.2}$$

$$S_2 \;=\; \exp[-\frac{i}{2}(\phi_1+\phi_2)]\ , \tag{5.3}$$

[7] We will use NSR formalism in the following. It should be also possible to do this in the GS formalism by using the methods of Ref. [11] where the topological nature of these amplitudes should become more transparent.

Similarly the spin fields $S^{\dot\alpha}$ for the anti-chiral spacetime fermions $\bar\chi$ have the bosonized expressions:

$$S^{\dot 1} \;=\; \exp[\frac{i}{2}(\phi_1-\phi_2)]\ , \tag{5.4}$$

$$S^{\dot 2} \;=\; \exp[-\frac{i}{2}(\phi_1-\phi_2)]\ , \tag{5.5}$$

As already mentioned, we also bosonize the β,γ system in terms of a free boson φ and the η,ξ system as usual [12].

By ghost charge counting, and recalling that on a genus g Riemann surface there are also $2g-2$ additional insertions of picture-changing operators, we have that the total number of insertions of picture-changing operators is $3g-3+n$.

The anti-self-dual part of the gauge field vertex in the 0 ghost picture contains the following left-moving fermionic combinations: $\psi_1\psi_2$, $\bar\psi_1\bar\psi_2$ and $\psi_1\bar\psi_1+\psi_2\bar\psi_2$ depending on the kinematics. We will see below following the eq.(5.17) that the purely bosonic part of the gauge field vertices, $\partial X^\mu \bar J^a$, does not contribute to the amplitude under consideration.

To make the calculation transparent we choose a kinematic configuration such that g–1 of the λ's appear with S_1 and the remaining $g-1$ λ's with S_2 . Furthermore we take for left-moving part of one gauge field vertex $\psi_1\psi_2$, and for the second one $\bar\psi_1\bar\psi_2$. Also, we choose n of the $\bar\chi$'s with $S^{\dot 1}$ and the remaining n with $S^{\dot 2}$.

It will be also convenient to take the n $\bar\chi$'s appearing with $S^{\dot 2}$ in the unintegrated form by inserting n c-ghost fields. Correspondingly, we will have $3g-3+n$ b-ghost insertions to agree with the dimension of the moduli space of genus g Riemann surfaces with n punctures. Thus we are led to considering the following amplitude:

$$\begin{aligned} A^{g,n} \;=\;& \langle \prod_{k=1}^{3g-3+n} |(\mu_k b)|^2 \prod_{i=1}^{g-1} e^{-\frac{1}{2}\varphi} S_1\Sigma(x_i)\bar J^{a_i}(\bar x_i) \prod_{i=1}^{g-1} e^{-\frac{1}{2}\varphi} S_2\Sigma(y_i)\bar J^{b_i}(\bar y_i) \\ & \psi_1\psi_2(z)\bar J^{a_g}(\bar z)\bar\psi_1\bar\psi_2(w)\bar J^{b_g}(\bar w)\prod_{i=1}^{n} V^{\dot 1}_{\bar\imath_i}(u_i,\bar u_i) \\ & \prod_{i=1}^{n} c\bar c V^{\dot 2}_{\bar\jmath_i}(v_i,\bar v_i) \prod_{i=1}^{3g-3+n} e^{\varphi}T_F(z_i)\rangle_g\ , \end{aligned} \tag{5.6}$$

where the indices $\bar{\imath}$ and $\bar{\jmath}$ label the anti-modulini $\bar{\chi}$. We will show now that this amplitude is proportional to the topological amplitude $F^g_{\bar{\imath}_1\cdots\bar{\imath}_n;\bar{\jmath}_1\cdots\bar{\jmath}_n}$ for general $(2,0)$ compactifications.

The universal feature of all $(2,0)$ compactifications is the underlying $U(1)$ current algebra which can be bosonized in terms of a free scalar field H [13]. H is a compact boson and its momenta sit in a one-dimensional lattice given by the $U(1)$ charges of the states. Besides the part of the superconformal field theory containing the space-time fermionic coordinates and the superghosts, the spin structure dependence enters only through appropriate shifts of the one-dimensional lattice of H. The remaining part of the internal theory does not see the spin structures. Therefore to do the spin structure sum it is sufficient to know how it enters in the $U(1)$-charge lattice. On the other hand the topological theory involves precisely twisting by adding an appropriate background charge for the field H, and again the rest of the internal theory is insensitive to this twisting. It is this fortunate circumstance that will enable us to show the equivalence between the string amplitude and $A^g_{\bar{\imath}_1\cdots\bar{\imath}_n;\bar{\jmath}_1\cdots\bar{\jmath}_n}$, without ever needing to know the details of the compactification.

Let Γ be the $U(1)$ lattice of H momenta. The space-time fermionic coordinates define an $SO(2)\times SO(2)$ lattice. If one takes one of these $SO(2)$ lattices and combines it with Γ, then it is known that the resulting 2-dimensional lattice is given by the coset $E_6/SO(8)$ [14, 15]. The characters are given by the branching functions $F_{\Lambda,s}(\tau)$ satisfying:

$$\chi_\Lambda(\tau)=\sum_s F_{\Lambda,s}(\tau)\chi_s(\tau)\ , \tag{5.7}$$

where χ_Λ and χ_s are E_6 and $SO(8)$ level one characters, Λ denotes the three classes of E_6, and we are using the spin structure basis denoted by s to represent the four conjugacy classes of $SO(8)$. The characters of the internal conformal field theory times one complex space-time fermionic coordinate can then be represented as $F_{\Lambda,s}(\tau)\mathrm{Ch}_\Lambda(\tau)$, where $\mathrm{Ch}_\Lambda(\tau)$ is the contribution of the rest of the internal theory. The essential point here is that $\mathrm{Ch}_\Lambda(\tau)$ depends only on Λ and not on the $SO(8)$ representations or equivalently on the spin structures. Generalization of this to higher genus is obtained by assigning an E_6

representation Λ for each loop and we will denote this collection by $\{\Lambda\}$.

To proceed further, we note that in the amplitude (5.6), due to the conservation of $U(1)$ charge only G^- parts of the T_F's contribute. As before, let us consider the contribution of the left-moving sector to the amplitude. The dependence of G^- on H is given by:

$$G^- = e^{-iH/\sqrt{3}}\hat{G}^- \ , \tag{5.8}$$

where $\hat{G}^-$ has no singular operator product expansion with the $U(1)$ current and it carries a dimension 4/3. Furthermore, the anti-chiral fields $\Psi_{\bar{\imath}}$ carry charge (-1), and therefore their H-dependence is given by:

$$\Psi_{\bar{\imath}} = e^{-iH/\sqrt{3}}\hat{\Psi}_{\bar{\imath}}, \tag{5.9}$$

where $\hat{\Psi}$ has non singular OPE with the $U(1)$ current and has dimension 1/3. It follows that the internal part $\Sigma_{\bar{\imath}}$ of the fermion vertex (3.9) is given by:

$$\Sigma_{\bar{\imath}} = e^{+iH/2\sqrt{3}}\hat{\Psi}_{\bar{\imath}}. \tag{5.10}$$

Taking the specific kinematic configuration as in (5.6), one can now explicitly compute the correlation functions for the space-time fermions, superghost and the free field H. After combining the lattice of (say) ψ_2 with that of H, the result can be expressed as:

$$\begin{aligned} A^{g,n}_{(f)} \quad = \quad & \frac{\theta_s(\frac{1}{2}\sum_i(x_i-y_i)+\frac{1}{2}\sum_k(u_k-v_k)+z-w)}{Z_1\theta_s(\frac{1}{2}\sum_i(x_i+y_i)+\frac{1}{2}\sum_k(u_k+v_k)-\sum_a z_a+2\Delta)} \\ & F_{\{\Lambda\},s}(\frac{1}{2}\sum_i(x_i-y_i)-\frac{1}{2}\sum_k(u_k-v_k)+z-w\ ; \\ & \frac{\sqrt{3}}{2}\sum_i(x_i+y_i)+\frac{1}{2\sqrt{3}}\sum_k(u_k+v_k)-\frac{1}{\sqrt{3}}\sum_a z_a) \\ & \frac{\prod_{i<j}E(x_i,x_j)E(y_i,y_j)}{E^2(z,w)\prod_{a<b}E^{2/3}(z_a,z_b)}\prod_i\frac{E(x_i,z)E(y_i,w)}{E(x_i,w)E(y_i,z)} \\ & \frac{\prod_{k<\ell}E^{1/3}(u_k,u_\ell)E^{1/3}(v_k,v_\ell)}{\prod_{k,\ell}E^{2/3}(u_k,v_\ell)}\prod_{k,a}E^{1/3}(u_k,z_a)E^{1/3}(v_k,z_a) \\ & \frac{\prod_i\sigma(x_i)\sigma(y_i)\prod_k\sigma(u_k)\sigma(v_k)}{\prod_a\sigma^2(z_a)}G_{\{\Lambda\}}(\{z_a,u_k,v_k\})\ , \end{aligned} \tag{5.11}$$

where θ_s denotes the genus g θ-function of spin structure s, E is the prime form and Z_1 is the chiral determinant of the $(1,0)$ system. Δ is the Riemann θ-constant, which represents the degree $g-1$ divisor of a half differential associated with the preferred spin structure for a given marking of the Riemann surface. $\sigma(x)$ is a $g/2$-differential with no zeros or poles, transforming in a quasiperiodic way under a_i or b_i monodromies. Finally, the function $G_{\{\Lambda\}}$ is given by:

$$G_{\{\Lambda\}}(\{z_a,u_k,v_k\}) = \langle \prod_a \hat{G}^-(z_a) \prod_k \hat{\Psi}_{\bar{\imath}_k}(u_k)\hat{\Psi}_{\bar{\jmath}_k}(v_k) \prod_i Q_k^{a_i}\bar{\omega}_k(\bar{x}_i) \prod_j Q_k^{b_j}\bar{\omega}_k(\bar{y}_j)\rangle_{\{\Lambda\}} \ , \quad (5.12)$$

where i and j run from 1 to g, with $x_g = z$ and $y_g = w$, and k runs over the g anti-holomorphic differentials. $G_{\{\Lambda\}}$ represents the contribution of the internal conformal field theory after removing the H contribution and does not depend on spin structure. As explained in section 2 we have replaced the Kac-Moody currents by their zero-mode parts. $F_{\{\Lambda\},s}(u;v)$ represents the $SO(2)\times\Gamma$ lattice contribution to genus g partition function with sources u and v; u is coupled to $SO(2)$ lattice and v is coupled to Γ [14].

To do the sum over spin structures we choose the positions z_a such that

$$\sum_a z_a = \sum_i y_i + \sum_k v_k - z + w + 2\Delta. \quad (5.13)$$

As a result the theta functions in the numerator of eq.(5.11), cancel with those in the denominator. The only spin structure dependence then appears in $F_{\{\Lambda\},s}$. We can sum over the spin structures by using the formula [14]:

$$\sum_s F_{\{\Lambda\},s}(u;v) = F_{\{\Lambda\}}(\frac{1}{2}u + \frac{\sqrt{3}}{2}v \ ; \ \frac{\sqrt{3}}{2}u - \frac{1}{2}v) \ , \quad (5.14)$$

where

$$F_{\{\Lambda\}}(u;v) = \theta(u)\Theta_{\{\Lambda\}}(v) \ . \quad (5.15)$$

$\Theta_{\{\Lambda\}}$ are given by:

$$\Theta_{\{\Lambda\}}(v) = \sum_{n_i\in\mathbf{Z}} exp(3\pi i(n_i + \frac{\lambda_i}{3})\tau_{ij}(n_j + \frac{\lambda_j}{3}) + 2\pi i\sqrt{3}(n_i + \frac{\lambda_i}{3})v_i) \ , \quad (5.16)$$

where $i,j = 1\cdots g$, and $\lambda_i = 0,1,2$ depending on E_6 conjugacy class Λ_i corresponding to **1**, **27** and $\overline{\mathbf{27}}$, respectively. They can be expressed as combinations of level six theta functions of Refs. [14, 15]. Using this formula, (5.11) becomes:

$$\begin{aligned} A^{g,n}_{(f)} \quad = \quad & \frac{\theta(\sum_i x_i + z - w)\Theta_{\{\Lambda\}}(\frac{1}{\sqrt{3}}(\sum_a z_a + \sum_k(u_k - 2v_k) - 3\Delta))}{Z_1} \\ & \frac{\prod_{i<j} E(x_i,x_j)E(y_i,y_j)}{E^2(z,w)\prod_{a<b} E^{2/3}(z_a,z_b)} \prod_i \frac{E(x_i,z)E(y_i,w)}{E(x_i,w)E(y_i,z)} \\ & \frac{\prod_{k<\ell} E^{1/3}(u_k,u_\ell)E^{1/3}(v_k,v_\ell)}{\prod_{k,\ell} E^{2/3}(u_k,v_\ell)} \prod_{k,a} E^{1/3}(u_k,z_a)E^{1/3}(v_k,z_a) \\ & \frac{\prod_i \sigma(x_i)\sigma(y_i)\prod_k \sigma(u_k)\sigma(v_k)}{\prod_a \sigma^2(z_a)} G_{\{\Lambda\}}(\{z_a,u_k,v_k\}) \ . \end{aligned} \tag{5.17}$$

Note that the contribution from the bosonic part of the gauge vertices, namely $\partial X^\mu \bar{J}^a$, would be the same as above except that the argument of the first theta function on the right hand side becomes just $\sum_i x_i$ and hence the expression vanishes due to Riemann vanishing theorem.

Using the bosonization formula for untwisted spin $(1,0)$ and $(2,-1)$ determinants and using the condition (5.13), we see that all the space-time fermion boson as well as ghost and superghost non-zero mode determinants cancel. We get finally:

$$\begin{aligned} A^{g,n}_{(f)} \quad = \quad & \frac{\prod_{a<b} E^{1/3}(z_a,z_b)\prod_{k<\ell} E^{4/3}(v_k,v_\ell)}{\prod_{k,a} E^{2/3}(v_k,z_a)} \frac{\prod_{k,a} E^{1/3}(u_k,z_a)\prod_{k<\ell} E^{1/3}(u_k,u_\ell)}{\prod_{k,\ell} E^{2/3}(u_k,v_\ell)} \\ & \frac{\prod_a \sigma(z_a)\prod_k \sigma(u_k)}{\prod_\ell \sigma^2(v_\ell)} \frac{\Theta_{\{\Lambda\}}(\frac{1}{\sqrt{3}}(\sum_a z_a + \sum_k(u_k - 2v_k) - 3\Delta))}{\det h_a(z_b)} \\ & G_{\{\Lambda\}}(\{z_a,u_k,v_k\}) \frac{\det \omega_i(x_j,z)\det \omega_i(y_j,w)}{(\det \mathrm{Im}\tau)^2} \\ & |\det(\mu_a h_b)|^2 \text{ (right-moving part) } . \end{aligned} \tag{5.18}$$

Here h_a denote a basis of $3g-3+n$ quadratic differentials, dual to μ_a, which span the cotangent space to $M_{g,n}$. They are allowed to have simple poles at the punctures v_k. One can check that the expression in (5.18) has the correct conformal properties, namely it transforms as a quadratic differential in each z_a, as a zero form in v_k and as a $1-$form in

u_k. As in the case of F^g discussed in section 2, the factor $(\det \mathrm{Im}\tau)^{-2}$ is canceled after integrating over z, w, x_i and y_i.

Consider the factor of (5.18) which depends on z_a:

$$
\begin{aligned}
B(\{z_a,\}) \quad &\equiv \quad \frac{\prod_{a<b} E^{1/3}(z_a,z_b) \prod_a \sigma(z_a) \prod_{k,a} E^{1/3}(u_k,z_a)}{\prod_{k,a} E^{2/3}(v_k,z_a)} \\
&\quad \Theta_{\{\Lambda\}}(\frac{1}{\sqrt{3}}(\sum_a z_a + \sum_k (u_k - 2v_k) - 3\Delta))G_{\{\Lambda\}}(\{z_a,u_k,v_k\}) \ . \qquad (5.19)
\end{aligned}
$$

It transforms as a quadratic differential in each z_a, it is holomorphic with first order zeroes as $z_a \to z_b$ for $a \neq b$ (generically), since $G_{\{\Lambda\}}(\{z_a,u_k,v_k\})$ goes as $(z_a - z_b)^{2/3}$ for $z_a \to z_b$, and it is totally antisymmetric in z_a. It has also simple poles as $z_a \to v_k$, since $G_{\{\Lambda\}}(\{z_a,u_k,v_k\})$ goes as $(z_a - v_k)^{-1/3}$ for $z_a \to v_k$. This implies that

$$
B(\{z_a\}) \propto \det h_a(z_b) \ , \qquad (5.20)
$$

with proportionality constant independent of z_a. So, we can "bring under the integral" of $\det(\mu_a h_b)$ the ratio $B(\{z_a\})/\det h_a(z_b)$ appearing in (5.18). Integration over x_i, y_i, z and w gives $(g!) \cdot (\det Im\tau)^2 \det Q_k^{a_i} \det Q_k^{b_j}$. The final result is

$$
\begin{aligned}
A_{(f)}^{g,n} \quad &= \quad \int_{M_{g,n}} \frac{\prod_{k<\ell} E^{4/3}(v_k,v_\ell) \prod_{k<\ell} E^{1/3}(u_k,u_\ell) \prod_k \sigma(u_k)}{\prod_{k,\ell} E^{2/3}(u_k,v_\ell) \prod_\ell \sigma^2(v_\ell)} \\
&\quad \int \prod_{a=1}^{3g-3+n} d^2 z_a (\prod_{a=1}^{3g-3+n} \mu_a B(\{z\}) \prod_{a=1}^{3g-3+n} \bar{\mu}_a \bar{b}(\bar{z}_a)) \\
&\quad \det Q_k^{a_i} \det Q_k^{b_j} (\text{right-moving part}) \ . \qquad (5.21)
\end{aligned}
$$

Here in the right moving part we put all the contributions coming from the bosonic sector which have not been explicitly displayed.

Now let us compare this result with the correlation functions of the topological theory as given in (3.6). Once again we extract the H dependence from G_- and from the fields $\Psi_{\bar{\imath}}$, $\tilde{\Psi}_{\bar{\jmath}}$, of charges -1 and 2 respectively, and explicitly work out its correlation function, remembering that in the twisted version there is a background charge $\frac{\sqrt{3}}{2} \int R^{(2)} H$ in the

action. $3g-3+n$ G_-'s appearing in the topological partition function that are folded with the Beltrami differentials precisely balance this background charge plus the net charge of Ψ's and $\tilde{\Psi}$'s, which is n. One easily finds that $F^g_{\bar{\imath}_1,\dots,\bar{\imath}_n;\bar{\jmath}_1,\dots,\bar{\jmath}_n} = A^{g,n}/(g!)^2$. This completes the proof that the heterotic string amplitudes under consideration are proportional to the topological amplitudes of (3.6) for general $(2,0)$ compactifications.

6. Effective Field Theory

In this section we discuss relations between the genus g topological amplitudes $F^g_{\bar{\imath}_1\cdots\bar{\imath}_n;\bar{\jmath}_1\cdots\bar{\jmath}_n}$ and the physical scattering amplitudes of heterotic superstring theory. We begin by giving a superfield description of the underlying effective lagrangian interactions. These interactions can be understood as higher-dimensional F-terms. The procedure that we adopt here is to construct such terms in superconformal supergravity, and to impose subsequently the standard N=1 Poincaré gauge-fixing constraints [16].

In superconformal supergravity, the multiplets are characterized by their Weyl and chiral weights (ω,ν) which specify the properties under the dilatations and chiral $U(1)$ transformations. Under complex conjugation, $(\omega,\nu)\to(\omega,-\nu)$. The chiral superfields containing moduli and matter fields carry weights (0,0). One additional chiral superfield, the compensator Σ with weights (1,1), is introduced for the sole purpose of breaking the superconformal group down to N=1 Poincaré supersymmetry, by constraining its scalar and fermionic components. An invariant action can be constructed from the F-component of a chiral superfield with weights (3,3). For example, the standard superpotential w is a function of (0,0) fields only, and the corresponding lagrangian density is obtained from the F-component of $\Sigma^3 w$; in this case, the (3,3) weights are supplied entirely by the chiral compensator. Also, the gauge kinetic terms can be written as F-terms involving the square of the canonical gauge field-strength W carrying weights $(3/2,3/2)$; the (3,3) weights are then supplied by W^2.

Higher-dimensional F-terms can be constructed by including superfields which are chiral projections of complex vector superfields. The superconformal chiral projection Π, which is a generalization of the $\bar{D}^2$ operator of rigid supersymmetry, can be defined for vector superfields V with weights (2,0) only; a chiral superfield $\Pi(V)$ has weights (3,3) [16]. It is convenient to define [7]

$$\Pi_\alpha \;\equiv\; \Pi(\Sigma\overline{\Sigma}\, e^{-K/3} f^\alpha)\,, \tag{6.1}$$

where K is the tree-level Kähler potential and f^α are arbitrary functions of chiral and/or anti-chiral (0,0) superfields. The factor $e^{-K/3}$ has been introduced to ensure covariant transformation properties of Π's under the Kähler transformations $K \to K + \varphi + \bar{\varphi}$, $\Sigma \to e^{\varphi/3}\Sigma$; Π's transform then with the same holomorphic weights as the defining f's. Let us introduce a class of chiral $(3,3)$ superfields

$$\mathcal{I}^g_n \;=\; F^g_n \cdot \Sigma^{3(1-g-n)} \cdot \Pi_1\Pi_2\cdots\Pi_n \cdot W^{2g}\ , \tag{6.2}$$

where F^g_n are arbitrary *analytic* functions of $(0,0)$ chiral superfields. We will see that the topological amplitudes are related to the interactions obtained from F-terms of such superfields. Note that Peccei-Quinn symmetry of the effective action forbids the dilaton-dependence of F^g_n, with the exception of tree-level gauge kinetic terms contained in F^1_0.

In order to determine the string loop order of $\mathcal{I}^g_n$, the powers of the string coupling constant should be counted in the following way. In the superconformal theory, the Planck mass squared M_P^2 corresponds to the vacuum expectation value of the scalar component of $\Sigma\overline{\Sigma}e^{-K/3}$. The gauge fixing constraint on the scalar component of Σ that results in a sigma-model type normalization $M_P^2 \sim 1/g_s^2$, with g_s the four-dimensional string coupling, corresponds to $\Sigma \sim e^{K/6}g_s^{-1}$.From the known dilaton-dependence of K it follows that $K \sim \ln g_s^2$, therefore $\Sigma \sim g_s^{-2/3}$. Assuming that the functions F^g_n and f^α do not depend on the dilaton, we have $\Pi_\alpha \sim g_s^{-2}$ and $\mathcal{I}^g_n \sim g_s^{2g-2}$. Hence $\mathcal{I}^g_n$ describe g-loop string interactions.

To obtain explicit expressions for particle interactions, we will write down the component content of superfields Σ and Π. In general, a chiral superfield $\Phi = (z_\Phi, \mathcal{Z}_\Phi, \mathcal{F}_\Phi)$,

where z, $\mathcal{Z}$ and $\mathcal{F}$ are the scalar, fermionic and auxiliary components, respectively. The multiplication rule is

$$(z_1, \mathcal{Z}_1, \mathcal{F}_1) \times (z_2, \mathcal{Z}_2, \mathcal{F}_2) = (z_1 z_2,\ z_1\mathcal{Z}_2 + z_2\mathcal{Z}_1,\ z_1\mathcal{F}_2 + z_2\mathcal{F}_1 - 2\mathcal{Z}_1\mathcal{Z}_2). \tag{6.3}$$

After superconformal gauge fixing,[8] the compensator becomes $\Sigma = e^{K/6}(1, \mathcal{Z}_\Sigma, \mathcal{F}_\Sigma)$, with

$$\mathcal{Z}_\Sigma = \frac{1}{3} K_i \chi^i \tag{6.4}$$

$$\mathcal{F}_\Sigma = h^0 + \frac{1}{3}K_i(h^i + \Gamma^i_{jk}\chi^j\chi^k) - \frac{1}{3}(K_{ij} + \frac{1}{3}K_iK_j)\chi^i\chi^j \ . \tag{6.5}$$

Here, χ^i and h^i denote the fermionic and auxiliary components, respectively, of the physical moduli and matter fields. h^0 is an auxiliary field left over from the compensator; it is usually eliminated from the effective lagrangian together with all other auxiliary fields by using their equations of motion. The reparametrization connection $\Gamma^k_{ij} \equiv K^{-1\,k\bar{l}}K_{ij\bar{l}}$.

The components of $\Pi(\Sigma\overline{\Sigma}\, e^{-K/3} f)$ are [7]:

$$z_\Pi = \bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\nabla_{\bar{\imath}} f_{\bar{\jmath}} - h^{\bar{0}} f - h^{\bar{\imath}} f_{\bar{\imath}} \tag{6.6}$$

$$\begin{aligned} \mathcal{Z}_\Pi &= f_{\bar{\imath}\bar{\jmath}}\not{\partial} z^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} + f_{\bar{\imath}}\not{\partial}\bar{\chi}^{\bar{\imath}} \\ &\quad + \frac{f}{3}(K_{i\bar{\jmath}}\not{\partial} z^i\bar{\chi}^{\bar{\jmath}} - 2\sigma^{mn}\partial_m\psi_n)h^{\bar{0}} f_i\chi^i - \frac{1}{3}(K_{\bar{\imath}j} f - 3f_{\bar{\imath}j})h^{\bar{\imath}}\chi^j + \ldots \end{aligned} \tag{6.7}$$

$$\begin{aligned} \mathcal{F}_\Pi &= -f_{\bar{\imath}\bar{\jmath}}\partial_m z^{\bar{\imath}}\partial_m z^{\bar{\jmath}} - f_{\bar{\imath}}\partial^2 z^{\bar{\imath}} - \frac{f}{6}(2K_{i\bar{\jmath}}\partial_m z^i\partial_m z^{\bar{\jmath}} + R) \\ &\quad - h^0 h^{\bar{0}} f + \frac{1}{3}(K_{\bar{\imath}j} f - 3f_{\bar{\imath}j})h^{\bar{\imath}}h^j - f_{\bar{\imath}}h^{\bar{\imath}}h^0 - f_i h^i h^{\bar{0}} \\ &\quad + \{h^0\nabla_{\bar{\imath}} f_{\bar{\jmath}}\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} + h^k\nabla_{\bar{\imath}} f_{\bar{\jmath}k}\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} - \frac{2}{3}h^k\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}K_{k\bar{\imath}} f_{\bar{\jmath}} + \text{c.c.}\} + \ldots \end{aligned} \tag{6.8}$$

where ψ_m is the gravitino field. Here, the reparametrization covariant derivative ∇ acts as $\nabla_{\bar{\imath}} f_{\bar{\jmath}} = f_{\bar{\imath}\bar{\jmath}} - \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} f_{\bar{k}}$. The terms neglected in eqs.(6.7) and (6.8) are not relevant to the following discussion. We complete our list by recalling the component content of the gauge kinetic multiplet:

$$W^2 = [\ \lambda\lambda,\ F_{mn}\sigma^{mn}\lambda + \ldots,\ \frac{1}{2}(F_{mn}F^{mn} + iF_{mn}\tilde{F}^{mn}) + \ldots\ ]\ , \tag{6.9}$$

[8]From now on we set the string coupling $g_s = 1$.

where λ and F_{mn} are the gaugino and gauge field strength, respectively. Here again, we omitted some irrelevant terms.

The F-term lagrangian density obtained from a (3,3) chiral superfield Φ is given by the standard formula [16]:

$$\Phi|_F = \mathcal{F}_\Phi + \bar{\psi}_m \bar{\sigma}^m \mathcal{Z}_\Phi + \bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n \, z_\Phi + \text{c.c.} \tag{6.10}$$

Note that the scalar component of Φ gives rise to a field-dependent gravitino mass. By using the expressions (6.4)-(6.9), the multiplication rule (6.3) and the above formula, one can derive explicit form of the interactions induced by $\mathcal{I}_n^g$ of eq.(6.2). It is worth mentioning that the tree-level kinetic energy terms, which are usually written as D-terms, can also be written as the F-term of

$$\mathcal{I}_1^0 = 3\,\Pi(\Sigma\overline{\Sigma}\, e^{-K/3}) \ , \tag{6.11}$$

c.f. eqs.(6.10) and (6.8).

The genus $g = 0$ topological amplitudes are related to the tree-level interactions $\mathcal{I}_n^0$, $n > 1$, in the following way. Let us consider the tree-level scattering amplitude

$$\left\langle \bar{\chi}^{\bar{\imath}_1}, \bar{\chi}^{\bar{\imath}_2}, \ldots, \bar{\chi}^{\bar{\imath}_{n-1}}; \bar{\chi}^{\bar{\jmath}_1}, \bar{\chi}^{\bar{\jmath}_2}, \ldots, \bar{\chi}^{\bar{\jmath}_{n-1}}; z^{\bar{\imath}_n}, z^{\bar{\jmath}_n} \right\rangle \ = \ p^{\bar{\imath}_n} \!\cdot\! p^{\bar{\jmath}_n} \, A^{0,n} \ , \tag{6.12}$$

where $A^{0,n}$ is a momentum-independent function of the background moduli fields. The helicity configuration is chosen in such a way that $(\bar{\chi}_1^{\bar{\imath}_\alpha}, \bar{\chi}_{\dot{2}}^{\bar{\imath}_\alpha}) = (\theta^{\bar{\imath}_\alpha}, 0)$ and $(\bar{\chi}_1^{\bar{\jmath}_\alpha}, \bar{\chi}_{\dot{2}}^{\bar{\jmath}_\alpha}) = (0, \eta^{\bar{\jmath}_\alpha})$. This amplitude can be generated by the interaction terms contained in $\mathcal{I}_n^0$ and $\mathcal{I}_m^0$, $m < n$. The effective interaction term that we are interested in has the form

$$e^{\frac{(1-n)}{2}K} F^0_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\ldots\bar{\jmath}_n} \theta^{\bar{\imath}_1}\theta^{\bar{\imath}_2}\ldots\theta^{\bar{\imath}_{n-1}}\eta^{\bar{\jmath}_1}\eta^{\bar{\jmath}_2}\ldots\eta^{\bar{\jmath}_{n-1}}\,\partial_p \bar{z}^{\bar{\imath}_n}\partial^p \bar{z}^{\bar{\jmath}_n} \ . \tag{6.13}$$

The functions $F^0_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\ldots\bar{\jmath}_n}$ are given by the tree-level topological amplitudes considered in the previous section.

Let us first consider

$$\mathcal{I}_n^0|_F \ = \ F_n^0 \cdot \Sigma^{3(1-n)} \cdot \Pi_1\Pi_2\cdots\Pi_n|_F \ . \tag{6.14}$$

$\mathcal{I}^0_n|_F$ gives rise to two basic contributions to eq.(6.13). The first one is due to the terms of the form $z_{\Pi_1} z_{\Pi_2} \dots \mathcal{F}_{\Pi_k} \dots z_{\Pi_n}$, with $\mathcal{F}_\Pi$ and z_Π's contributing $\partial\bar{z}\partial\bar{z}$ and fermion bilinears, respectively, see eqs.(6.6) and (6.8). The second one is due to the terms of the form $z_{\Pi_1} z_{\Pi_2} \dots \mathcal{Z}_{\Pi_k} \mathcal{Z}_{\Pi_l} \dots z_{\Pi_n}$, with both $\mathcal{Z}_\Pi$'s contributing $\not{\partial}\bar{z}\bar{\chi}$, see eq.(6.7), and z_Π's as before. After Fierz transformation, this contribution acquires the desired form of eq.(6.13). The physical amplitudes receive also additional contributions due to reducible diagrams with scalar and/or fermion propagators attached to the operators $\partial^2\bar{z}$ and $\not{\partial}\bar{\chi}$ contained in $\mathcal{F}_\Pi$'s and $\mathcal{Z}_\Pi$'s; after including these diagrams, the effective action becomes manifestly field-reparametrization invariant. Combining everything together yields

$$F^0_{\bar{\imath}_1\bar{\imath}_2\dots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\dots\bar{\jmath}_n} = F^0_n \sum_{\sigma,\omega} (-1)^{\mathrm{sgn}(\sigma)+\mathrm{sgn}(\omega)}\, f^1_{\bar{\imath}_{\sigma(1)}\bar{\jmath}_{\omega(1)}} f^2_{\bar{\imath}_{\sigma(2)}\bar{\jmath}_{\omega(2)}} \dots f^n_{\bar{\imath}_{\sigma(n)}\bar{\jmath}_{\omega(n)}} \ , \tag{6.15}$$

where the summation extends over all permutations σ and ω of n indices $\bar{\imath}$ and $\bar{\jmath}$, respectively. As a result, F^0 is completely antisymmetric in $\bar{\imath}$ and $\bar{\jmath}$ indices separately. It also satisfies trivially the identity (3.7). Furthermore, it satisfies

$$\nabla_{[\bar{\imath}} F^0_{\bar{\imath}_1\bar{\imath}_2\dots\bar{\imath}_n];\bar{\jmath}_1\bar{\jmath}_2\dots\bar{\jmath}_n} = 0 \ , \tag{6.16}$$

due to Kähler geometry and analyticity of F^0_n, as required by supersymmetry.

Before explaining how $\mathcal{I}^0_m$, with $m<n$, affect $2n$-point topological amplitudes we would like to point out another class of related scattering amplitudes. As already mentioned before, see eq.(6.10), locally supersymmetric F-terms give rise to field-dependent gravitino mass terms. These contribute to the amplitude

$$\left\langle \bar{\chi}^{\bar{\imath}_1}, \bar{\chi}^{\bar{\imath}_2}, \dots, \bar{\chi}^{\bar{\imath}_n}; \bar{\chi}^{\bar{\jmath}_1}, \bar{\chi}^{\bar{\jmath}_2}, \dots, \bar{\chi}^{\bar{\jmath}_n}; \bar{\psi}_{p\,\dot{\alpha}}, \bar{\psi}^{\dot{\beta}}_q \right\rangle \ = \ \bar{\sigma}^{pq\,\dot{\alpha}}{}_{\dot{\beta}}\, M^{0,n} \ , \tag{6.17}$$

where $M^{0,n}$ is a momentum-independent function of the background moduli fields. Here again, the helicity configuration is chosen in the same way as in the amplitude (6.12). The corresponding effective action term has the form

$$\tfrac{1}{(n!)^2} e^{\frac{(1-n)}{2}K} F^0_{\bar{\imath}_1\bar{\imath}_2\dots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\dots\bar{\jmath}_n} \theta^{\bar{\imath}_1}\theta^{\bar{\imath}_2}\dots\theta^{\bar{\imath}_n}\eta^{\bar{\jmath}_1}\eta^{\bar{\jmath}_2}\dots\eta^{\bar{\jmath}_n}\ \bar{\psi}_p\bar{\sigma}^{pq}\bar{\psi}_q \ . \tag{6.18}$$

The fact that the same (topological) functions F^0 appear above as in the scalar-fermion interactions of eq.(6.13) is a direct consequence of supersymmetry. It is easy to see that $\mathcal{I}_n^0|_F$ gives rise to an effective gravitino mass of the form (6.18), with the coefficient given by eq.(6.15).

The scattering amplitudes of eqs.(6.12) and (6.17), and hence the respective $2n$-point topological amplitudes, receive also contributions from the interactions induced by $\mathcal{I}_m^0$ with $m < n$. In general, these interactions arise from reducible diagrams with the auxiliary h-fields propagating on internal lines; these may combine with diagrams that are reducible in the physical scalar and fermion lines. Let us illustrate this point by constructing a typical contribution to the fermion-scalar amplitude (6.12). $\mathcal{I}_m^0|_F$ induces an interaction term of the form $\partial_i F_m^0\, h^i(\bar{\chi}\bar{\chi})^m$. The auxiliary field h^i can propagate to the vertex $h^{\bar{\imath}}(\partial\bar{z}\partial\bar{z})(\bar{\chi}\bar{\chi})^{m'-2}$ induced by (possibly the same) $\mathcal{I}_{m'}^0|_F$, producing an effective $(\bar{\chi}\bar{\chi})^{m+m'-2}(\partial\bar{z}\partial\bar{z})$ coupling which has the same structure as the one induced directly by $\mathcal{I}_n^0|_F$ with $n = m + m' - 1$. This is only one of quite a few reducible diagrams. Note also that in general, due to the presence of higher power auxiliary field interactions, $h\bar{h}^m$ $(m > 1)$, their elimination can be highly non-trivial. In Appendix A, we show explicitly how to evaluate $\mathcal{I}_3^0 \sim \Pi^3$-type terms induced by $\mathcal{I}_2^0|_F \sim \Pi^2$, however it is not clear how to proceed in more complicated cases.

To recapitulate, the computation of the effects of higher-dimensional F-terms on the scattering amplitudes involves elimination of auxiliary fields. As a result, the complete functions $F^0_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\ldots\bar{\jmath}_n}$ no longer satisfy eq.(6.16). Instead, they satisfy the recursion relations (3.10) (in which $g = 0$) obtained in section 3 by using topological methods which capture also, in a simple way, the effects of complicated field-theoretical diagrams. These relations are not sufficient though to determine all functions since the solutions can only be determined up to holomorphic pieces that satisfy the homogeneous eq.(6.16) as discussed in section 4. Thus the analytic coefficients F_n^0 of Π^n terms remain arbitrary, whereas non-holomorphic parts are constrained by eqs.(3.10). We are not able to prove here rigorously

that the elimination of auxiliary fields in Π^n-type terms yields precisely the same equations. The example discussed in Appendix A provides some circumstantial evidence that this is indeed the case. Further support is provided by the interpretation of the tree-level recursion relation as a supersymmetric Ward identity (3.16).

In section 4, we presented an algorithm to generate non trivial solutions of the recursion relations. It employs a Ward identity associated to the transformations (4.5). From the supergravity point of view, these transformations correspond to the residual supersymmetry in a special type of superconformal gauge, $\Sigma = (1, 0, h^0)$. In this gauge, the standard supersymmetry transformation $\delta_Q(\epsilon)$ [16] must be followed by a compensating S supersymmetry transformation $\delta_S(\zeta = \epsilon h^0)$ which is necessary to restore the gauge. Such a combined transformation which preserves the form of the superconformal gauge fixing condition defines Poincaré supersymmetry. The transformation $\delta_s(\varepsilon)$ (4.5) corresponds to "1/2" of Poincaré supersymmetry, namely with $\epsilon = 0$, $\bar{\epsilon} = (0, \bar{\varepsilon})$ and with all spacetime derivative terms set to zero. It is not surprising that this transformation is not nilpotent on the fields, since the anticommutator of Q and S generators produces additional dilatations and axial rotations.

Although from the topological point of view, eq.(3.10) describes an anomaly, there is clearly no field-theoretical anomaly at the tree-level. Such an anomaly, the so-called holomorphic anomaly, appears though in one-loop threshold corrections to gauge couplings. It shows up in higher genus recursion relations, as explained below.

The higher genus F-terms, $\mathcal{I}^g_n \sim F^g_n \Pi^n W^{2g}$, can be discussed in a similar way. In sections 2 and 3 we have considered genus g amplitudes

$$\left\langle \bar{\chi}^{\bar{\imath}_1}, \ldots, \bar{\chi}^{\bar{\imath}_n}; \bar{\chi}^{\bar{\jmath}_1}, \ldots, \bar{\chi}^{\bar{\jmath}_n}; \lambda^{a_1}_{\alpha_1}, \lambda^{b_1}_{\beta_1}, \ldots, \lambda^{a_{g-1}}_{\alpha_{g-1}}, \lambda^{b_{g-1}}_{\beta_{g-1}}; F^{a_g}_{mn}, F^{b_g}_{pq} \right\rangle$$

$$= A^{g,n}\, \delta_{mp}\delta_{nq}\, \delta^{a_g b_g} \prod_{k=1}^{g-1} \delta^{a_k b_k} \epsilon_{\alpha^k \beta^k} \qquad (6.19)$$

with the usual helicity configuration of $\bar{\chi}$'s. The corresponding effective action term,

$$e^{\frac{(1-g-n)}{2}K} g F^{g}_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\ldots\bar{\jmath}_n} \theta^{\bar{\imath}_1}\theta^{\bar{\imath}_2}\ldots\theta^{\bar{\imath}_n}\eta^{\bar{\jmath}_1}\eta^{\bar{\jmath}_2}\ldots\eta^{\bar{\jmath}_n}\,(\lambda\lambda)^{g-1}\,F_{mn}F^{mn}\ , \tag{6.20}$$

can be induced either directly by $\mathcal{I}^g_n$ or via reducible diagrams involving couplings induced by $\mathcal{I}^{g'}_{n'}$ with $n' < n$ and/or $g' < g$. The one-loop threshold correction F^1 receives also contributions from the anomalous, non-local effective action terms that violate holomorphicity of the field-dependent gauge couplings. These anomalies can feed into higher genus through reducible diagrams, as illustrated in Appendix B, where a one-loop $\mathcal{I}^1_1 \sim \Pi W^2$-type term is induced by a tree-level $\mathcal{I}^0_2|_F \sim \Pi^2$ interaction, via one-loop, anomalous W^2 couplings. Also in the next section we show by an explicit string computation on a simple orbifold example, that a tree-level Π^3 interaction generates a sequence of non-holomorphic terms: $\Pi^2 W^2$ at the one-loop, ΠW^4 at two loops, and W^6 at three loops.

We should finally mention that as in the tree-level case, the F-term interactions (6.20) have locally supersymmetric completions which include field-dependent gravitino mass terms,

$$e^{\frac{(1-g-n)}{2}K} F^{g}_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n;\bar{\jmath}_1\bar{\jmath}_2\ldots\bar{\jmath}_n} \theta^{\bar{\imath}_1}\theta^{\bar{\imath}_2}\ldots\theta^{\bar{\imath}_n}\eta^{\bar{\jmath}_1}\eta^{\bar{\jmath}_2}\ldots\eta^{\bar{\jmath}_n}\,(\lambda\lambda)^{g}\,\bar{\psi}_p\bar{\sigma}^{pq}\bar{\psi}_q\ . \tag{6.21}$$

As discussed in Sections 2 and 3, from the topological point of view, the recursion relations (3.10) reflect anomalies of the underlying two-dimensional, twisted superconformal field theory. These anomalies are also reflected in the scattering amplitudes of massless superstring excitations in four dimensions. Here, within the framework of effective field theory, these relations describe an intricate interplay within a large class of higher-dimensional F-terms (6.2) and can be understood as a consequence of $N = 1$ Poicaré supersymmetry.

7. Orbifold examples

Here we will work out some simple examples in orbifold models of the topological quantities we have introduced in sections 2 and 3.

In the case of orbifolds, the internal $N=2$ SCFT is realized in terms of free bosons and fermions. We consider for simplicity orbifolds realized in terms of 3 complex bosons X_I and left-moving fermions Ψ_I, with $I=1,2,3$, together with 16 right-moving bosons living on the $E_8\times E_8$ lattice. Let h be an element of the orbifold group defined by $h=\{h_I,\delta_h\}$, and its action on X_I is $X_I\to e^{2\pi i h_I}X_I$ and similarly for Ψ_I and δ_h denotes a shift on the 16 right-moving bosons. Space-time supersymmetry implies that one can always choose the h_I's to satisfy the condition:

$$\sum_I h_I \;=\; 0\ . \tag{7.1}$$

On a genus g Riemann surface we must associate to each homology cycle a_i,b_i, for $i=1,\cdots,g$, an element of the orbifold group. In the following we shall denote by $\{h\}=\{\{h_I,\delta_h\}\}$ the set of all twists along different cycles. One can bosonize the complex fermions

$$\Psi_I=e^{i\Phi_I}\ ,\qquad\qquad \overline{\Psi}_I=e^{-i\Phi_I}\ , \tag{7.2}$$

The previously introduced free boson H (see section 5) can be expressed in terms of the Φ_I's as: $\sqrt{3}H=\sum_{I=1}^3\Phi_I$. We also need the form of vertex operators for the untwisted anti-moduli $\overline{T}_I$:

$$V_{\overline{T}_I}=(\partial\overline{X}_I+ip\cdot\psi\overline{\Psi}_I)\bar\partial X_I e^{ip\cdot X}\ . \tag{7.3}$$

In the case of a $\mathbf{Z}_2$ twisted plane, there is also an associated U-modulus. The $\overline{U}_I$ vertex operator $V_{\overline{U}_I}$ is obtained from the above expression by replacing $\bar\partial X^I$ with $\bar\partial\overline{X}^I$. Similarly, the vertex for an untwisted $\bar\chi_{\overline{T}_I}$ is given by:

$$V^{\dot\alpha}_{\overline{T}_I}=(e^{-\frac{\varphi}{2}}S^{\dot\alpha}e^{-\frac{\Phi_I}{2}+\frac12\sum_{J\neq I}\Phi_J})(z)\bar\partial X^I(\bar z)e^{ip\cdot X}\ , \tag{7.4}$$

with a similar expression for $V^{\dot\alpha}_{\overline{U}_I}$.

At the tree-level, the simplest topological quantity is of the type Π^2 which was considered in Ref. [7]. However, it was shown there to be zero for untwisted moduli whereas for twisted moduli it was shown to be non-vanishing in an explicit example of $\mathbf{Z}_6$ orbifold in Ref. [17]. For untwisted moduli, the simplest non-vanishing topological quantity is of the

type Π^3. This corresponds to a 6-point amplitude (6.12) involving two anti-moduli (7.3) and four anti-modulini (7.4). Using the explicit form of the vertices and left-moving charge conservation, one can easily see that a non-vanishing contribution is obtained only for $\mathbf{Z}_2$ twists and when both T- and U-type fields are present for all three planes. Moreover, the vertices of the two anti-moduli contribute only with their fermionic part providing the two space-time momenta. In an appropriate kinematic configuration, the result can be recast in the form of the topological amplitude:

$$F^0_{\overline{T}_1\overline{T}_2\overline{T}_3;\overline{U}_1\overline{U}_2\overline{U}_3} =$$
$$\langle\bar\partial X_1\overline{\Psi}_1(z_1)\bar\partial X_2\overline{\Psi}_2(z_2)\bar\partial X_3\overline{\Psi}_3(z_3)\bar\partial\overline{X}_1\Psi_2\Psi_3(w_1)\bar\partial\overline{X}_2\Psi_3\Psi_1(w_2)\bar\partial\overline{X}_3\Psi_1\Psi_2(w_3)\rangle\ , \tag{7.5}$$

where Ψ_I and $\overline{\Psi}_I$ ($I = 1,2,3$) are, respectively, zero- and one-forms in the topological theory. Note that the fermion bilinears $\Psi_I\Psi_J$ appearing in the expression (7.5) are precisely the left moving parts of the (left) charge 2 and dimension (0,0) operators $\tilde\Psi_{\bar\jmath}$ defined in eq.(3.6) with the holomorphic three-form $\rho = \Psi_1\Psi_2\Psi_3$. The final result is:

$$F^0_{\overline{T}_1\overline{T}_2\overline{T}_3;\overline{U}_1\overline{U}_2\overline{U}_3} = \prod_{I=1}^{3}\frac{1}{(T_I+\overline{T}_I)^2(U_I+\overline{U}_I)^2}\ , \tag{7.6}$$

while all other non-vanishing amplitudes can be obtained by using the antisymmetry properties of F^0 and the symmetry under interchanging T- and U-moduli of the same plane. Notice that this expression is covariantly constant with respect to all 6 moduli consistently with the absence of a Π^2 term.

Eq.(7.6) can be integrated by using eq.(6.15) to obtain the functions $f^{1,2,3}$ which enter into the $\Pi_1\Pi_2\Pi_3$ interaction term. However, this integration is not unique [7] because all physical amplitudes involve at least two anti-analytic derivatives of each f. A consistent solution is that $F^0_3 = 1$ and that f^I depends only on the T and U moduli of the I-th plane with

$$f^I_{\overline{T}_I\overline{U}_I} = \frac{1}{(T_I+\overline{T}_I)^2(U_I+\overline{U}_I)^2}\ . \tag{7.7}$$

One can now use the $SL(2,Z)^6$ duality symmetry associated with the 6 moduli to integrate eq.(7.7). $SL(2,Z)$ symmetries induce Kähler transformations which can be compensated by transforming f's. It is easy to see that Π_I must transform as a form of weight 2 with respect to the $SL(2,Z)$'s corresponding to T_I and U_I. This implies that

$$f^I = G_2(T_I)G_2(U_I) \tag{7.8}$$

where $G_2(T) \equiv 1/(T+\overline{T}) + 2\partial_T \ln\eta(iT)$. Note that the analytic part of G_2 drops from all physical tree-level amplitudes consistently with the Peccei-Quinn symmetries of the untwisted moduli.

To discuss examples of topological terms of the type $W^{2g}\Pi^n$ we will consider models having at least two different pure gauge groups with no massless matter representations, so that one does not get contributions to the recursion relations from handle degeneration. A simple way to obtain such models using fermionic constructions of $\mathbf{Z}_2\times\mathbf{Z}_2$ orbifolds is by introducing more than one sets of periodic right-moving fermions [18]. Each set should contain a number of real fermions which must be multiple of 8 due to world-sheet modular invariance. In this way, with two such sets, one gets from the Neveu-Schwarz sector gauge bosons in the adjoint representation of $O(8k)\times O(8l)$ (with $k+l\leq 5$) and no massless states in vector representations. For the simplest choice, $k=3$ and $l=2$, we obtain $O(24)\times E_8$ with no massless matter. However in this case it is easy to see that there are only two moduli associated to one of the 3 internal planes. To construct families of models with at least 3 untwisted moduli associated with 3 different planes one should choose for instance $k=3$ and $l=1$. In this case, a potential problem is the generic appearance of massless $O(8)$ spinors coming from the corresponding Ramond sector. However, we checked in several examples that these states can be eliminated by an appropriate choice of GSO projections among the various sets of the basis vectors, leading to a pure gauge group $O(24)\times O(8)$.

Going back to the 6 moduli example we considered before, one sees from the structure

of the recursion relations (3.10) that a Π^3 term on the sphere generates a $W^2\Pi^2$ term on the torus, a $W^4\Pi$ term on genus 2 and a W^6 term on genus 3 with non holomorphic couplings. In the following, we will present a string derivation of these terms and check the form of the recursion relations.

Let us start from the $W^2\Pi^2$ term at genus 1. We can, for instance, take the highest component of the Π^2 superfield times the lowest component of W^2: this will lead us to consider an amplitude involving two gauginos, two anti-moduli and two anti-modulini. It is easy to see, by left-moving charge conservation, that the two anti-moduli and two anti-modulini should come from two different planes. We will take the anti-moduli to be $\overline{T}_1$ and $\overline{U}_2$ and the anti-modulini to be the supersymmetric partners of $\overline{U}_1$ and $\overline{T}_2$. Choosing a convenient kinematic configuration, we are thus lead to evaluate the string amplitude:

$$A^{1,2} = \langle V_\lambda^1(x) V_\lambda^2(y) V_{\overline{U}_1}^{\dot{1}}(z) V_{\overline{T}_2}^{\dot{2}}(w) V_{\overline{T}_1}(u) V_{\overline{U}_2}(v) e^{\varphi} T_F(z_1) e^{\varphi} T_F(z_2)\rangle \ , \tag{7.9}$$

where V_λ^α is the gaugino vertex operator (2.6). Here we assume that the third plane is untwisted. Using internal left-moving charge conservation one can show that only the fermionic part of the anti-moduli vertices contribute to the amplitude providing also the two factors of space-time momenta. One can then set the momenta to zero in the remaining part of the vertices. The spin structures sum can be performed as usual by using the Riemann theta-identity. The resulting expression can be written as a topological amplitude:

$$F^1_{\overline{T}1,\overline{T}2;\overline{U}1,\overline{U}2} = \langle \bar\partial X_1 \overline{\Psi}_1(z) \bar\partial X_2 \overline{\Psi}_2(v) \bar\partial \overline{X}_2 \Psi_3 \Psi_1(w) \bar\partial \overline{X}_1 \Psi_2 \Psi_3(u) G^-(z_1) G^-(z_2) \delta Q^2\rangle \ , \tag{7.10}$$

where δQ^2 denotes the difference between the squared-charges of the two gauge groups. G^- is the dimension $(2,0)$ current $G^- = \sum_I \overline{\Psi}_I \partial X_I$ and by charge conservation we see that only the part corresponding to the third plane contributes to the amplitude we are considering. Furthermore since this plane is untwisted, ∂X_3 is replaced by its zero-mode p_L.

One can now perform the integrations over the positions of the vertices, using the theta-functions expression for the correlator of the scalars $X_{1,2}$ twisted by $g_{1,2} = g$, $\langle \bar\partial \overline{X}_{1,2} \bar\partial X_{1,2}\rangle$,

and doing partial integrations in the $\bar{z}$ and $\bar{v}$ variables. One is then left with a final integral of the square of the Szego kernel corresponding to the twist g, which can again be performed by a partial integration, giving a factor $\tau_2\partial_{\bar\tau}\ln(\tau_2\bar\theta_g)$, where θ_g is the theta-function with characteristics given by the twist g. The final expression for the amplitude is reduced to an integral over the moduli space of the torus:

$$\begin{aligned} F^1_{\overline{T}_1,\overline{T}_2;\overline{U}_1\overline{U}_2} &= \prod_{I=1,2}\frac{1}{(T_I+\overline{T}_I)^2(U_I+\overline{U}_I)^2} \\ &\times\int d^2\tau\tau_2\sum_{p_L,p_R}\frac{p_L^2}{(T_3+\overline{T}_3)(U+\overline{U}_3)}e^{i\pi\tau|p_L|^2}e^{i\pi\bar\tau|p_R|^2}\bar\eta(\bar\tau)^{-2}\mathrm{Tr}(\delta Q^2)\partial_{\bar\tau}\ln(\tau_2\bar\theta_g). \quad (7.11) \end{aligned}$$

Here the lattice sum extends over the $\Gamma^{2,2}$ lattice of the (untwisted) third plane. The dependence on the moduli of the first two planes is given by the explicit pre-factor multiplying the integral in (7.11) due to the normalization of the corresponding vertex operators.

In order to evaluate the r.h.s. of eq.(7.11), it is convenient to take its derivative with respect to $\overline{T}_3$. Using the identity [7]:

$$\begin{aligned} &\partial_{\overline{T}_3}\sum_{p_L,p_R}\frac{p_L^2}{(T_3+\overline{T}_3)(U+\overline{U}_3)}e^{i\pi\tau|p_L|^2}e^{i\pi\bar\tau|p_R|^2} \\ &\qquad\qquad = \frac{-i}{\pi\tau_2(T_3+\overline{T}_3)^2}\partial_\tau\big(\tau_2\partial_{U_3}\sum_{p_L,p_R}e^{i\pi\tau|p_L|^2}e^{i\pi\bar\tau|p_R|^2}\big)\ , \quad (7.12) \end{aligned}$$

we obtain, after a partial integration over τ,

$$\begin{aligned} \partial_{\overline{T}_3}F^1_{\overline{T}_1,\overline{T}_2;\overline{U}_1,\overline{U}_2} &= \frac{1}{(T_3+\overline{T}_3)^2}\prod_{I=1,2}\frac{1}{(T_I+\overline{T}_I)^2(U_I+\overline{U}_I)^2} \\ &\quad \partial_{U_3}\int\frac{d^2\tau}{\tau_2}\sum_{p_L,p_R}\bar\eta(\bar\tau)^{-2}\mathrm{Tr}(\delta Q^2)e^{i\pi\tau|p_L|^2}e^{i\pi\bar\tau|p_R|^2} \quad (7.13) \end{aligned}$$

Here we used the fact that, upon taking ∂_{U_3} derivative, the $\tau_2\to\infty$ boundary term vanishes for generic moduli values. The integral on the r.h.s. of eq.(7.13) is the familiar expression for the group-dependent part of threshold corrections to gauge couplings [19], proportional to $\ln[|\eta(iT_3)\eta(iU_3)|^4(T_3+\overline{T}_3)(U_3+\overline{U}_3)]$. Integrating eq.(7.13) we obtain

$$F^1_{\overline{T}_1,\overline{T}_2;\overline{U}_1\overline{U}_2} = \delta\tilde{b}\,G_2(T_3)G_2(U_3)\prod_{I=1,2}\frac{1}{(T_I+\overline{T}_I)^2(U_I+\overline{U}_I)^2}\ , \quad (7.14)$$

where $\delta\tilde{b}$ is the difference between the usual threshold function coefficients [19] of the two gauge groups. Note that eq.(7.13) has precisely the form of the recursion relation (3.10) for genus 1:

$$\partial_{\overline{T}_3} F^1_{\overline{T}_1,\overline{T}_2;\overline{U}_1,\overline{U}_2} = G^{U_3\overline{U}_3} F^0_{\overline{T}_1,\overline{T}_2,\overline{T}_3;\overline{U}_1,\overline{U}_2,\overline{U}_3} \partial_{U_3} F^1 \tag{7.15}$$

with

$$F^1 = \delta\tilde{b} \sum_{I=1}^{3} \ln[|\eta(iT_I)\eta(iU_I)|^4 (T_I + \overline{T}_I)(U_I + \overline{U}_I) \tag{7.16}$$

and the inverse metric $G^{U_3\overline{U}_3} = (U_3 + \overline{U}_3)^2$.

Eq. (7.14) shows that the coefficient

$$F^1_2(T_3, U_3) = \delta\tilde{b}\, G_2(T_3) G_2(U_3) \tag{7.17}$$

of the F-term $F^1_2 \Pi_1 \Pi_2 W^2$ suffers from a holomorphic anomaly analogous to the one appearing in threshold corrections i.e. coefficients of the one loop W^2 terms. The corresponding interactions should be in fact represented by non-local terms in the effective action. Of course, due to the complete symmetry among the three orbifold planes, similar anomalies occur in all other $\Pi^2 W^2$ terms.

At two loops, we expect holomorphic anomalies in ΠW^4 terms. Similarly to the one-loop case, we can take four gauginos from the lowest component of W^4 and two anti-moduli, say $\overline{T}_1$ and $\overline{U}_1$ from Π_1. Choosing a convenient kinematic configuration, we are lead to evaluate the string amplitude:

$$A^{2,1} = \langle V^1_\lambda(x) V^2_\lambda(y) V^1_\lambda(z) V^2_\lambda(w) V_{\overline{T}_1}(u) V_{\overline{U}_1}(v) \prod_{i=1}^{4} e^{\varphi} T_F(z_i)\rangle \ . \tag{7.18}$$

It is easy to see again by left-moving charge conservation that the vertex operators of the two anti-moduli contribute with their fermionic parts, providing also the two powers of momenta. The spin structure sum can be performed by the Riemann theta-identity after a convenient gauge choice of the position of picture changing operators along the lines of Ref. [4]. The result is given by the following topological amplitude:

$$F^2_{\overline{T}1;\overline{U}1} = \langle \bar{\partial} X_1 \overline{\Psi}_1(u) \bar{\partial} \overline{X}_1 \Psi_2 \Psi_3(v) \prod_{i=1}^{4} G^-(z_i) \delta(\det Q)^2 \rangle \tag{7.19}$$

where $\delta(\det Q)^2$ denotes an appropriate combination of products of determinants which ensures modular invariance, resulting from a combination of physical amplitudes in which there is no contribution from the contraction of the Kac-Moody currents (chosen in the Cartan subalgebra). Again, by using left-moving charge conservation we see that two G^-'s provide two factors of ∂X_2 and the other two provide two factors of ∂X_3. All of them are replaced by their zero-modes, given by 1-forms twisted by g_2 and g_3 respectively.[9] Similarly one can see that the fields $\bar{\partial} X_1$ and $\bar{\partial} \overline{X}_1$ also contribute with their zero mode part, given by the anti-holomorphic 1-form twisted by g_1.

Unlike in the one-loop case we are unable to compute $F^2_{\overline{T}1;\overline{U}1}$ directly however we will use its expected modular properties to write a solution of the recursion relations:

$$F^2_{\overline{T}1;\overline{U}1} = \frac{(\delta\tilde{b})^2}{(T_1+\overline{T}_1)^2(U_1+\overline{U}_1)^2} \prod_{I=2,3} G_2(T_I)G_2(U_I) \tag{7.20}$$

which indeed satisfies

$$\bar{\partial}_{\overline{T}_I} F^2_{\overline{T}1;\overline{U}1} = G^{U_I\overline{U}_I} F^1_{\overline{T}1,\overline{T}_I;\overline{U}1,\overline{U}_I} \partial_{U_I} F^1 \tag{7.21}$$

for $I = 2, 3$ and similarly for $T_I \leftrightarrow U_I$ interchange. We thus have

$$F^2_1 = (\delta\tilde{b})^2 \prod_{I=2,3} G_2(T_I)G_2(U_I) \tag{7.22}$$

corresponding to a term $F^2_1 \Pi^1 W^4$ in the effective action.

Finally at genus 3, we have to consider a W^6 term with non holomorphic coupling. The corresponding string amplitude involves two gauge bosons and four gauginos and it was computed in section 2. The resulting topological amplitude is given in eq.(2.10):

$$F^3 = \langle \prod_{i=1}^{6} G^-(z_i)\delta(\det Q)^2 \rangle \ . \tag{7.23}$$

Since there are two fermionic zero-modes for each plane, the 6 G^-'s provide 2 factors of $\bar{\partial} X_I$ for all three planes. As in genus 2 case we have the following ansatz for F^3:

$$F^3 = (\delta\tilde{b})^3 \prod_{I=1}^{3} G_2(T_I)G_2(U_I) \tag{7.24}$$

[9]By Riemann-Roch, at genus g there are $g-1$ twisted holomorphic one-forms, ω^i_h, for the twist h. Correspondingly, the bosonic zero-modes $\partial X = \sum_{i=1}^{g-1} p_i \omega^i_h$ for given momenta p_i.

which satisfies the recursion relation:

$$\bar{\partial}_{\overline{T}_I} F^3 = G^{U_I \overline{U}_I} F^2_{\overline{T}_I;\overline{U}_I} \partial_{U_I} F^1 \tag{7.25}$$

8. Conclusions

In this paper we have extended the earlier results on topological amplitudes appearing in type II strings to the case of the N=1 heterotic string. The analogues of the topological partition functions F^g's are the amplitudes corresponding to terms of the form W^{2g} in the effective action with W being the gauge superfields. The main difference with respect to the type II case however is that the recursion relations do not close within such terms. We found that amplitudes involving anti-chiral fields appear also in the recursion relations. We further identified these new terms as the ones corresponding to higher weight F-terms $W^{2g}\Pi^n$ where Π is a chiral superfield obtained by a chiral projection of a general superfield. We have also derived a set of consistent recursion relations obeyed by these terms in the simpler case when only the boundary terms coming from the degeneration along dividing geodesics contribute. This is the case for models which do not involve massless charged matter fields. In particular we have analyzed in detail the tree level terms of the form Π^n and have given an algorithm to solve the recursion relations based on a finite dimensional path integral. Some explicit orbifold examples both at the tree level as well as at higher loop orders were also discussed.

We have also argued that these recursion relations are just the supersymmetric Ward identities on the generating function of all the connected graphs which is the object that string theory amplitudes compute. This interpretation is strongly supported by the field theory analysis described in the two appendices, as well as by the solution of the recursion relations presented in section 4.

The main open question left unanswered in this paper is the structure of recursion

relations when there are also contributions from the handle degenerations. When there are charged massless matter fields one would expect them to appear in the handle degeneration weighted by their charge squares. However including such a term in the recursion relation does not satisfy the integrability condition. A possible problem in this could be due to the singularities in the correlation functions when the charged matter fields are present and interact with the gauge fields.

It is possible to write down a consistent set of recursion relations:

$$\theta^{\bar{i}} D_{\bar{i}} F^g = G^{j\bar{j}} \Big(\sum_{g_1+g_2=g} \frac{\partial F^{g_1}}{\partial \eta^{\bar{j}}} D_j F^{g_2} + D_j \frac{\partial F^{g-1}}{\partial \eta^{\bar{j}}} \Big) \tag{8.1}$$

where the second term on the right hand side would be the contribution from the handle degeneration. Notice that the second term above is not weighted by charge squares. We believe therefore that this equation describes gravitational terms of the form $W^{2g}\Pi^n$ where W now is the gravitation chiral superfield (more precisely one should consider suitable differences between gravitational terms and gauge terms that involve no charged massless matter fields). Given our interpretation of the recursion relations as expressing supersymmetry Ward identity, the second term above should be seen as quantum correction to the latter.

The higher weight F-terms discussed in this paper are expected to play important role for supersymmetry breaking. In particular, in the presence of gaugino (or other fermionic) condensates induced by non perturbative effects, they will generate a dynamical superpotential for the dilaton and moduli fields. They could also be useful for testing duality conjectures for various $N = 1$ theories [23], in a similar way as $N = 2$ F-terms (i.e. F_g's) have already provided non-trivial tests [21] of type II - heterotic duality for $N = 2$ superstring compactifications [22].

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Appendix A

In this appendix we illustrate the field-theoretic calculation mentioned in section 6. We start by adding to the standard, minimal $N = 1$ supergravity Lagrangian [20] a higher weight F-term of the type Π^2:

$$\begin{aligned} L_{\Pi^2} &= e^{-K/2}(\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\nabla_{\bar{\imath}}f^{(1}_{\bar{\jmath}} - h^{\bar{\imath}}f^{(1}_{\bar{\imath}})\,(f^{2)}_{\bar{k}\bar{l}}\partial_m z^{\bar{k}}\partial_m z^{\bar{l}} + f^{2)}_{\bar{k}}\partial^2 z^{\bar{k}}) \\ &\quad + e^{-K/2}(f^{(1}_{\bar{\imath}\bar{\jmath}}\,\not{\partial} z^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} + f^{(1}_{\bar{\imath}}\,\not{\partial}\bar{\chi}^{\bar{\imath}})\,(f^{2)}_{\bar{k}\bar{l}}\,\not{\partial} z^{\bar{k}}\bar{\chi}^{\bar{l}} + f^{2)}_{\bar{k}}\,\not{\partial}\bar{\chi}^{\bar{k}}) + \ldots, \end{aligned} \tag{A.1}$$

where a round bracket on the superscript denotes symmetrization. As explained in section 6, in the topological theory this term corresponds to an amplitude $F^0_{\bar{\imath}_1\bar{\imath}_2;\bar{\jmath}_1\bar{\jmath}_2}$ which is closed, that is $\nabla_{[\bar{k}}F^0_{\bar{\imath}_1\bar{\imath}_2];\bar{\jmath}_1\bar{\jmath}_2} = 0$ as can be explicitly verified from (A.1).

To perform the calculation, it is convenient to adopt a procedure which has the advantage of keeping the correct normalization of the Ricci scalar: we introduce Lagrange multipliers S^α, $\alpha = 1, 2$, which are chiral superfields, whose first component and spinor component are s^α and η^α respectively. Given the two functions f^α, $\alpha = 1, 2$ specifying the Π^2 term, we introduce a modified Kähler potential $\widetilde{K} = K - 3\ln(1 + \frac{1}{3}S_\alpha f^\alpha + c.c.)$. Here $S_\alpha \equiv c_{\alpha\beta}S^\beta$, where the matrix $c_{\alpha\beta}$ is defined to be equal to 1 for $\alpha \neq \beta$ and zero otherwise. We also add to the action a superpotential term $\frac{1}{2}S_\alpha S^\alpha = S^1 S^2$. It is then clear that the equations of motion for S^α set $S^\alpha = \Pi_\alpha$, where Π_α has been defined in section 6.

The relevant part of the supergravity Lagrangian is then:

$$e^{-1}L = -\frac{3}{2}[\Sigma\bar{\Sigma}e^{(-\frac{1}{3}\widetilde{K})}]_D + ([\frac{1}{2}\Sigma^3 S_\alpha S^\alpha]_F + c.c.) + \ldots. \tag{A.2}$$

Here the subscripts D and F denote the action formula for a real vector density and for chiral density, respectively [16]. The dots refer to ordinary superpotential contributions.

We can then solve the equations of motion for the components s^α and η^α perturbatively in the number of f's and $\bar\chi$'s. Formally, we have extended the set of ordinary chiral (anti-chiral) superfields Z^i, $(Z^{\bar\imath})$ by adding two more chiral (anti-chiral) superfields S^α $(\bar S^{\bar\alpha})$. As a result, all the geometrical quantities entering in the supergravity Lagrangian, like metrics, connections, curvatures, being computed from $\widetilde{K}$, will have components along the two additional directions. The Z directions will be denoted by latin indices, the S directions with (lower case) greek indices.

Starting from this supergravity Lagrangian, we will then compute the full connected amplitude $\langle(\bar\chi)^4(\bar z)^2\rangle$ to the second order in the external momenta. We will see that there are two types of kinematic structures arising: $(\bar\chi^{\bar\imath},\bar\chi^{\bar\jmath})(\bar\chi^{\bar k}\bar\chi^{\bar\ell})\partial_\mu z^{\bar m}\partial_\mu z^{\bar n}$ and $(\bar\chi^{\bar\imath}\bar\sigma^{\mu\nu}\bar\chi^{\bar\jmath})(\bar\chi^{\bar k}\bar\chi^{\bar\ell})$ $\partial_\mu z^{\bar m}\partial_\nu z^{\bar n}$. Both correspond in the topological theory to $F^0_{\bar\imath_1\bar\imath_2\bar\imath_3;\bar\jmath_1\bar\jmath_2\bar\jmath_3}$, where the $\bar\imath$'s and $\bar\jmath$'s are partitions of $\bar\imath$, $\bar\jmath$, $\bar k$, $\bar\ell$, $\bar m$ and $\bar n$ which depend on the specific kinematic configuration in a way that will be explicitly discussed later. These amplitudes will be quartic in the f^α's.

As explained in section 6, *cf.* eq.(6.10), a superpotential term contributes to the "gravitino mass", $m_{3/2}$, i.e. to the term in the action proportional to $\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu$. That is in our case the Π^2 F-term gives rise to:

$$L_{\text{mass}} = e^{\widetilde{K}/2}\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu s^1 s^2 \ . \tag{A.3}$$

The lowest order solution of s_α equations of motion is:

$$s^{(0)}_\alpha = c_{\alpha\beta}e^{-K/2}\bar\chi^{\bar\imath}\bar\chi^{\bar\jmath}\nabla_{\bar\imath}\nabla_{\bar\jmath}f^\beta \ , \tag{A.4}$$

which yields the gravitino mass:

$$e^{-K/2}m^{(0)}_{3/2} = (\bar\chi^{\imath}\bar\chi^{\jmath})(\bar\chi^{\bar k}\bar\chi^{\bar\ell})(\nabla_{\bar\imath}\nabla_{\bar\jmath}f^1)\nabla_{\bar k}\nabla_{\bar\ell}f^2 \ . \tag{A.5}$$

For our purposes, we need to solve for s_α to the next order, $s^{(1)}_\alpha$, cubic in the f's. This is done by expanding in components (A.2), both from D and F terms and varying with

respect to $\bar{s}$. One gets:

$$\begin{aligned} s_\alpha^{(1)} &= \frac{1}{2}s_\alpha^{(0)}s_\beta^{(0)}f^\beta + \frac{1}{2}c_{\alpha\beta}f^\beta s_\gamma^{(0)}s^{\gamma(0)} + c_{\alpha\beta}\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}e^{-K/2}s_\gamma^{(0)}[-\frac{1}{3}\nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}(f^\beta f^\gamma) \\ &\quad + f_{\bar{k}}^\beta G^{\bar{k}\ell}(\nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}f_\ell^\gamma - \frac{1}{2}K_\ell\nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}f^\gamma)] \ . \end{aligned} \tag{A.6}$$

As for the spinor component η_α, it turns out we do not need the next to leading term, since it cancels out in the action. For the leading term, $\eta_\alpha^{(0)}$, we get, after expanding kinetic and potential terms and varying with respect to $\bar{\eta}$:

$$\eta_\alpha^{(0)} = ie^{-K/2}(f_{\alpha\bar{\imath}}(\bar{\chi}^{\bar{\imath}}\overleftarrow{\not{\mathcal{D}}}) + (\nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}f_\alpha)\bar{\chi}^{\bar{\imath}}\not{\partial}z^{\bar{\jmath}}). \tag{A.7}$$

Here $\mathcal{D}$ is the usual covariant derivative in $(\chi^\imath, \bar{\chi}^{\bar{\imath}})$ space. We do not need terms involving $\bar{s}$ and $\bar{\eta}$.

The next step is to substitute the results obtained back in the action, to get the relevant interactions. We will give some of the terms that enter into our computation. From the expansion of the bosonic kinetic terms we have:

$$\begin{aligned} L_{\mathrm{bk}} &= (f_{\alpha\bar{\jmath}}\Box z^{\bar{\jmath}} + \nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}f_\alpha\partial_\mu z^{\bar{\imath}}\partial_\mu z^{\bar{\jmath}})(s^{\alpha(0)} + s^{\alpha(1)}) \\ &\quad -\frac{1}{6}s^{\alpha(0)}s^{\beta(0)}[(f_\alpha f_\beta)_{\bar{\jmath}}\Box z^{\bar{\jmath}} + \nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}(f_\alpha f_\beta)\partial_\mu z^{\bar{\imath}}\partial_\mu z^{\bar{\jmath}}], \end{aligned} \tag{A.8}$$

where $\Box z^{\bar{\jmath}} \equiv \partial^2 z^{\bar{\jmath}} + \Gamma^{\bar{\jmath}}_{\bar{k},\bar{\imath}}\partial_\mu z^{\bar{k}}\partial_\mu z^{\bar{\imath}}$.

The same must be done for the fermionic part of the Lagrangian, involving η's and χ, $\bar{\chi}$'s:

$$\begin{aligned} L_{\mathrm{F}} &= 2if_{\alpha\bar{\imath}}(\bar{\chi}^{\bar{\imath}}\overleftarrow{\not{\mathcal{D}}})^{(0)}\eta^{\alpha(0)} - 2iG_{\alpha\bar{\imath}}(\bar{\chi}^{\bar{\imath}}\overleftarrow{\not{\mathcal{D}}})^{(1)}\eta^{\alpha(0)} \\ &\quad -2iG_{j\bar{\imath}}(\bar{\chi}^{\bar{\imath}}\overleftarrow{\not{\mathcal{D}}})^{(0)}\chi^j - 2iG_{j\bar{\imath}}(\bar{\chi}^{\bar{\imath}}\overleftarrow{\not{\mathcal{D}}})^{(1)}\chi^j \\ &\quad +e^{K/2}[\eta^{\alpha(0)}\eta^{\beta(0)}(e^{-s_\alpha^{(0)}f^\alpha/2}c_{\alpha\beta} + \widetilde{K}_\alpha s_\beta^{(0)} + \widetilde{K}_\beta s_\alpha^{(0)} - \Gamma^\gamma_{\alpha\beta}s_\gamma^{(0)}) \\ &\quad +2\eta_\alpha^{(0)}\chi^i(\widetilde{K}_i s_\alpha^{(0)} - \Gamma^\beta_{i\alpha}s_\beta^{(0)})] + \frac{1}{2}[G_{A\bar{B}}G_{C\bar{D}} - R_{A\bar{B}C\bar{D}}]^{(0)}\chi^A\chi^C\bar{\chi}^{\bar{B}}\bar{\chi}^{\bar{D}}. \end{aligned} \tag{A.9}$$

Here the superscripts (0) and (1) mean that the corresponding quantities are evaluated to zeroth and first order in s and in the last term capital indices run over both z (latin) and s (greek) indices. By definition, $\chi_\alpha \equiv \eta_\alpha$. In eqs.(A.9) and below we keep track of terms up to first order in χ^i which is sufficient for the purpose of our calculation.

After substituting in (A.9) the quantities (A.7) and (A.4), we get the desired expression for the interactions. There are two kinds of terms, those of zeroth order in χ^i, $L_{\rm F}^{(0)}$, which give rise to irreducible contributions, and those of first order, $L_{\rm F}^{(1)}$, giving rise to reducible diagrams.

$$\begin{aligned} L_{\rm F}^{(0)} \ = \ & e^{-K/2}[f_{\alpha\bar\imath}(\bar\chi^{\bar\imath}\overleftarrow{\not{\mathcal{D}}}) + (\nabla_{\bar\imath}\nabla_{\bar\jmath}f_\alpha)\bar\chi^{\bar\imath}\not\partial z^{\bar\jmath}][f_{\beta\bar\jmath}(\bar\chi^{\bar\jmath}\overleftarrow{\not{\mathcal{D}}}) + (\nabla_{\bar\jmath}\nabla_{\bar k}f_\beta)\bar\chi^{\bar\jmath}\not\partial z^{\bar k}] \\ & \times[c_{\alpha\beta} + e^{-K/2}(\frac{1}{3}(\nabla_{\bar m}\nabla_{\bar n}(f^\alpha f^\beta) - \frac{1}{2}f^\alpha_{\bar m}f^\beta_{\bar n} - 2f^\alpha\nabla_{\bar m}\nabla_{\bar n}f^\beta\frac{1}{2}c_{\alpha\beta}f_\gamma\nabla_{\bar m}\nabla_{\bar n}f^\gamma)] \\ & +2e^{-K}[f_{\alpha\bar\imath}(\bar\chi^{\bar\imath}\overleftarrow{\not{\mathcal{D}}}) + (\nabla_{\bar\imath}\nabla_{\bar\jmath}f_\alpha)\bar\chi^{\bar\imath}\not\partial z^{\bar\jmath}] \\ & \times[(\nabla_{\bar m}\nabla_{\bar n}f_\gamma)\bar\chi^{\bar m}\bar\chi^{\bar n}(\frac{1}{3}\nabla_{\bar\imath}(f^\alpha f^\gamma)(\bar\chi^{\bar\imath}\overleftarrow{\not{\mathcal{D}}}) + \frac{1}{3}\nabla_{\bar\imath}\nabla_{\bar\ell}(f^\alpha f^\gamma)\bar\chi^{\bar\imath}\not\partial z^{\bar\ell} - \frac{1}{4}f^\alpha_{\bar\imath}f^\gamma_{\bar\ell}\bar\chi^{\bar\imath}\not\partial z^{\bar\ell}) \\ & +\frac{1}{4}e^{K/2}f^\alpha_{\bar\imath}f^\gamma\bar\chi^{\bar\imath}\not\partial(e^{-K/2}\bar\chi^{\bar m}\bar\chi^{\bar n}\nabla_{\bar m}\nabla_{\bar n}f_\gamma)]. \end{aligned} \tag{A.10}$$

The terms of first order in χ^i (and second order in f), are given by:

$$\begin{aligned} L_{\rm F}^{(1)} \ = \ & -2ie^{-K/2}[f_{\alpha\bar\jmath}(\bar\chi^{\bar\jmath}\overleftarrow{\not{\mathcal{D}}}) + (\nabla_{\bar k}\nabla_{\bar\jmath}f_\alpha)\bar\chi^{\bar\jmath}\not\partial z^{\bar k})]\chi^i(\nabla_{\bar\imath}\nabla_{\bar\ell}f^\alpha_i - K_i\nabla_{\bar\imath}\nabla_{\bar\ell}f^\alpha - \frac{1}{2}f^\alpha_{\bar\ell}G_{i\bar\imath})\bar\chi^{\bar\imath}\bar\chi^{\bar\ell} \\ & -2ie^{-K/2}[f_{\alpha,i\bar\jmath}(\bar\chi^{\bar\jmath}\overleftarrow{\not{\mathcal{D}}}) + (\bar\chi^{\bar\jmath}\not\partial z^{\bar k})\nabla_{\bar\jmath}\nabla_{\bar k}f_{\alpha,i} \\ & -\frac{1}{4}G_{i\bar\jmath}\bar\chi^{\bar\jmath}(f_{\alpha,\bar k}\not\partial z^{\bar k} - f_{\alpha,k}\not\partial z^k)]\chi^i(\bar\chi^{\bar m}\bar\chi^{\bar n})\nabla_{\bar m}\nabla_{\bar n}f^\alpha \\ & -\frac{i}{2}G_{i\bar\jmath}\bar\chi^{\bar\jmath}f_\alpha\not\partial[\nabla_{\bar k}\nabla_{\bar\ell}(e^{-K/2}f^\alpha)\bar\chi^{\bar k}\bar\chi^{\bar\ell}]\chi^i. \end{aligned} \tag{A.11}$$

We are now in position to compute the amplitude $\langle\bar\chi^4\bar z^2\rangle$, to second order in external momenta. Let us start from amplitudes of the type $(\bar\chi^{\bar\imath}\bar\chi^{\bar\jmath})(\bar\chi^{\bar k}\bar\chi^{\bar\ell})\partial_\mu z^{\bar m}\partial_\mu z^{\bar n}$. It turns out that, for this case, (A.11) is the only term we need from the fermionic sector (A.9). We have, to begin with, irreducible contributions $A_{\rm irr}$ which can be read off directly from (A.8).

They are given by:

$$
\begin{aligned}
2e^K A_{\rm irr} \;\; = \;\; & -\frac{1}{3}(\nabla_{\bar m}\nabla_{\bar n}(f_\alpha f_\beta))(\nabla_{\bar i}\nabla_{\bar j}f^\alpha)\nabla_{\bar k}\nabla_{\bar \ell}f^\beta + (\nabla_{\bar m}\nabla_{\bar n}f_\alpha)f^\alpha(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta)(\nabla_{\bar i}\nabla_{\bar j}f_\beta) \\
& +(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta)f_\beta(\nabla_{\bar m}\nabla_{\bar n}f_\alpha)(\nabla_{\bar i}\nabla_{\bar j}f^\alpha) - \frac{2}{3}(\nabla_{\bar k}\nabla_{\bar \ell}f_\beta)(\nabla_{\bar m}\nabla_{\bar n}f_\alpha)(\nabla_{\bar i}\nabla_{\bar j}(f^\alpha f^\beta)) \\
& +2(\nabla_{\bar k}\nabla_{\bar \ell}f_\beta)(\nabla_{\bar m}\nabla_{\bar n}f_\alpha)f^\alpha_{\bar p}G^{i\bar p}(\nabla_{\bar i}\nabla_{\bar j}f^\alpha_i - \frac{1}{2}K_i\nabla_{\bar i}\nabla_{\bar j}f^\beta). \qquad \text{(A.12)}
\end{aligned}
$$

In addition to this irreducible contribution there are three reducible diagrams contributing to the above amplitude, as depicted in Figure 1. They are due to the presence of vertices in (A.8, A.11) and their contribution is given below.

From diagram 1A:

$$A_{1,\rm red.} = e^{-K}(\nabla_{\bar i}\nabla_{\bar j}f_\alpha)f^\alpha_{\bar p}G^{i\bar p}D_i[(\nabla_{\bar m}\nabla_{\bar n}f_\beta)(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta)]. \qquad \text{(A.13)}$$

From diagram 1B:

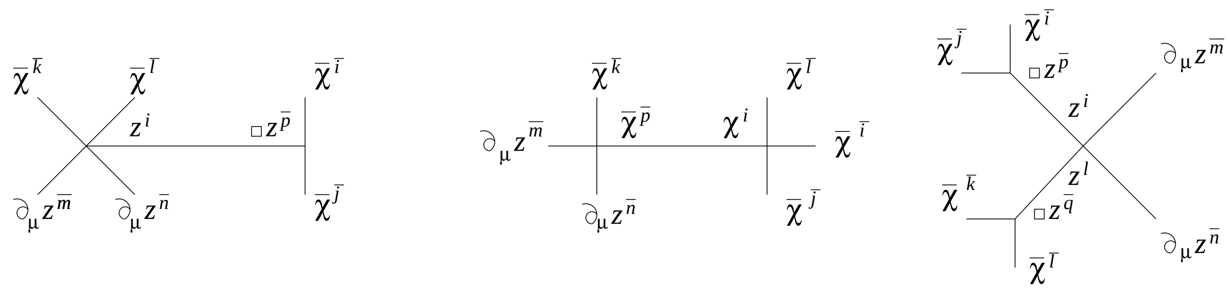
$$A_{2,\rm red.} = \frac{1}{2}e^{-K}f_\alpha(\nabla_{\bar i}\nabla_{\bar j}f^\alpha)(\nabla_{\bar m}\nabla_{\bar n}f_\beta)(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta). \qquad \text{(A.14)}$$

From diagram 1C:

$$A_{3,\rm red.} = \frac{1}{2}e^{-K}(\nabla_{\bar i}\nabla_{\bar j}f_\alpha)f^\alpha_{\bar p}(\nabla_{\bar k}\nabla_{\bar \ell}f_\beta)f^\beta_{\bar q}G^{i\bar p}G^{\ell\bar q}R_{i\bar m\ell\bar n}. \qquad \text{(A.15)}$$

In (A.13) D_i is the Kähler covariant derivative $\partial_i - K_i$. Adding all four contributions, we get finally for the total amplitude $A_{\rm tot}$:

$$
\begin{aligned}
2e^K A_{\rm tot} \;\; = \;\; & -\frac{1}{3}(\nabla_{\bar m}\nabla_{\bar n}(f_\alpha f_\beta))(\nabla_{\bar i}\nabla_{\bar j}f^\alpha)(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta) \\
& +(\nabla_{\bar m}\nabla_{\bar n}f_\alpha)f^\alpha(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta)(\nabla_{\bar i}\nabla_{\bar j}f_\beta) \\
& +(\nabla_{\bar m}\nabla_{\bar n}f_\alpha)f^\alpha_{\bar p}G^{i\bar p}D_i[(\nabla_{\bar i}\nabla_{\bar j}f_\beta)(\nabla_{\bar k}\nabla_{\bar \ell}f^\beta)] \\
& +R_{i\bar m\ell\bar n}(\nabla_{\bar i}\nabla_{\bar j}f_\alpha)f^\alpha_{\bar p}(\nabla_{\bar k}\nabla_{\bar \ell}f_\beta)f^\beta_{\bar q}G^{i\bar p}G^{\ell\bar q} + \text{cyclic permutations.} \qquad \text{(A.16)}
\end{aligned}
$$



1A 1B 1C

Figure 1: Reducible contributions to the kinematical structure $(\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}})(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}})\partial_\mu z^{\bar{m}}\partial_\mu z^{\bar{n}}$.

The cyclic permutations act on the set of pairs $(\bar{\imath}\bar{\jmath}),(\bar{k}\bar{\ell}),(\bar{m}\bar{n})$.

In a similar fashion one can study the kinematical structure $(\bar{\chi}^{\bar{\imath}}\not\partial z^{\bar{m}})(\bar{\chi}^{\bar{\jmath}}\not\partial z^{\bar{n}})(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}})$, which contributes both to the previous amplitude and to $(\bar{\chi}^{\bar{\imath}}\bar{\sigma}^{\mu\nu}\bar{\chi}^{\bar{\jmath}})(\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}})\partial_\mu z^{\bar{m}}\partial_\nu z^{\bar{n}}$, as it follows from the Fierz identity:

$$2(\bar{\chi}^{\bar{\imath}}\not\partial z^{\bar{m}})(\bar{\chi}^{\bar{\jmath}}\not\partial z^{\bar{n}}) = (\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}})\partial_\mu z^{\bar{m}}\partial_\nu z^{\bar{n}} + (\bar{\chi}^{\bar{\imath}}\bar{\sigma}^{\mu\nu}\bar{\chi}^{\bar{\jmath}})\partial_\mu z^{\bar{m}}\partial_\nu z^{\bar{n}}. \tag{A.17}$$

Again one has irreducible terms as well as (four) reducible diagrams. The irreducible contribution comes from (A.10). The reducible diagrams, shown in Figure 2 are obtained taking into account also the vertices of (A.11).

Putting everything together one arrives at the following result for this amplitude $B_{\rm tot}$:

$$\begin{aligned} e^K B_{\rm tot} \;=\;& -\frac{1}{3}(\nabla_{\bar{k}}\nabla_{\bar{\ell}}(f_\alpha f_\beta))(\nabla_{\bar{\imath}}\nabla_{\bar{m}}f^\alpha)\nabla_{\bar{\jmath}}\nabla_{\bar{n}}f^\beta) \\ & +(\nabla_{\bar{k}}\nabla_{\bar{\ell}}f_\alpha)f^\alpha(\nabla_{\bar{\imath}}\nabla_{\bar{m}}f^\beta)(\nabla_{\bar{\jmath}}\nabla_{\bar{n}}f_\beta) \\ & +(\nabla_{\bar{k}}\nabla_{\bar{\ell}}f_\alpha)f^\alpha_{\bar{p}}G^{i\bar{p}}D_i[(\nabla_{\bar{\imath}}\nabla_{\bar{m}}f_\beta)(\nabla_{\bar{\jmath}}\nabla_{\bar{n}}f^\beta)] \\ & +R_{i\bar{k}\ell\bar{\ell}}(\nabla_{\bar{\imath}}\nabla_{\bar{m}}f_\alpha)f^\alpha_{\bar{p}}(\nabla_{\bar{\jmath}}\nabla_{\bar{n}}f_\beta)f^\beta_{\bar{q}}G^{i\bar{p}}G^{\ell\bar{q}} + \text{cyclic permutations.} \end{aligned} \tag{A.18}$$

The cyclic permutations here act on the set of pairs $(\bar{\imath}\bar{m}),(\bar{k}\bar{\ell}),(\bar{\jmath}\bar{n})$. In principle, in both

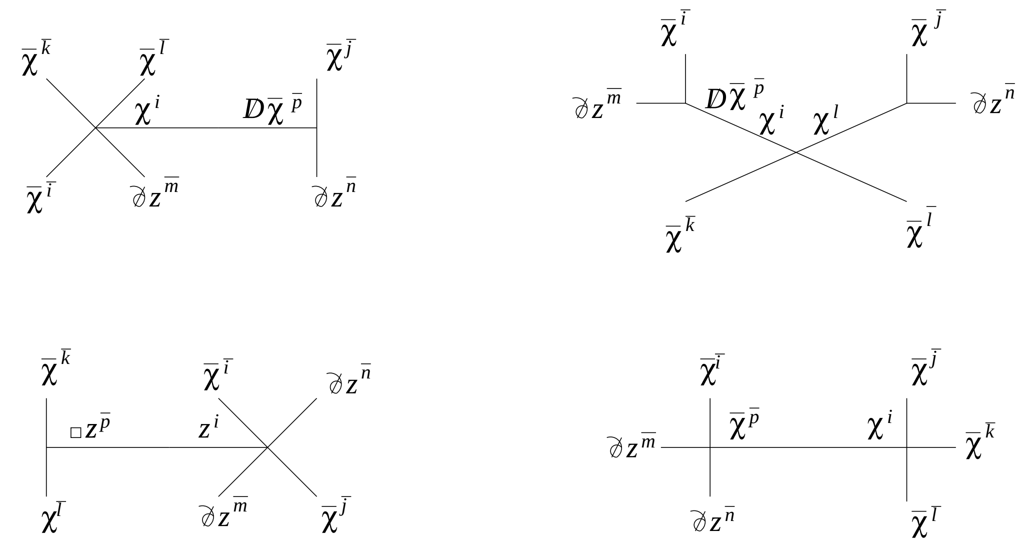


Figure 2: Reducible contributions to the kinematical structure $(\bar{\chi}^{\bar{\imath}} \not{\partial} z^{\bar{m}})(\bar{\chi}^{\bar{\jmath}} \not{\partial} z^{\bar{n}})(\bar{\chi}^{\bar{k}} \bar{\chi}^{\bar{\ell}})$.

cases, one could expect to have contributions coming from exchanges of gravitons or gravitinos. We have actually checked that they do not contribute to the amplitudes we are considering.

In view of (A.17) we can now compute the final expression for the two independent kinematic structures, i.e. the one symmetric in (μ, ν), the $\delta_{\mu\nu}$ term, coming from (A.16) and from (A.18), and that antisymmetric, containing $\bar{\sigma}^{\mu\nu}$, which comes from (A.18) only.

In either case, to make contact with the topological correlators, one has to consider terms with a given configuration of spinorial indices. Consider first the $\delta_{\mu\nu}$ term: taking, say, $\bar{\imath}$, $\bar{k}$ with spinorial component $\dot{\alpha} = 1$ and $\bar{\jmath}$, $\bar{\ell}$ with $\dot{\alpha} = 2$, we see that $(\bar{\imath}, \bar{k}, \bar{n})$ as well as $(\bar{\jmath}, \bar{\ell}, \bar{m})$ are totally antisymmetric. Similarly, for the $\bar{\sigma}_{\mu\nu}$ term, one can take $(\bar{\imath}, \bar{\jmath}, \bar{k})$ with $\dot{\alpha} = 1$ and $\bar{\ell}$ with $\dot{\alpha} = 2$ and one sees that again $(\bar{\imath}, \bar{\jmath}, \bar{k})$ and $(\bar{\ell}, \bar{m}, \bar{n})$ are totally antisymmetric.

In either case we have an amplitude $F^0_{\bar{\imath}_1\bar{\imath}_2\bar{\imath}_3;\bar{\jmath}_1\bar{\jmath}_2\bar{\jmath}_3}$, which has the form:

$$F^0_{\bar{\imath}_1\bar{\imath}_2\bar{\imath}_3;\bar{\jmath}_1\bar{\jmath}_2\bar{\jmath}_3} = (\nabla_{\bar{\imath}_1}\nabla_{\bar{\jmath}_1} f_\alpha) f^\alpha_{\bar{p}} G^{i\bar{p}} D_i[(\nabla_{\bar{\imath}_2}\nabla_{\bar{\jmath}_2} f_\beta)(\nabla_{\bar{\imath}_3}\nabla_{\bar{\jmath}_3} f^\beta)]$$

$$
\begin{aligned}
&+(\nabla_{\bar\imath_1}\nabla_{\bar\jmath_1}f_\alpha)f^\alpha_{\bar p}(\nabla_{\bar\imath_2}\nabla_{\bar\jmath_2}f_\beta)f^\beta_{\bar q}G^{\imath\bar p}G^{\ell\bar q}R_{\imath\bar\imath_3\ell\bar\jmath_3}\\
&+(\nabla_{\bar\imath_1}\nabla_{\bar\jmath_1}f_\alpha)f^\alpha(\nabla_{\bar\imath_2}\nabla_{\bar\jmath_2}f_\beta)(\nabla_{\bar\imath_3}\nabla_{\bar\jmath_3}f^\beta)\\
&-\frac{1}{3}(\nabla_{\bar\imath_1}\nabla_{\bar\jmath_1}(f^\alpha f^\beta))(\nabla_{\bar\imath_2}\nabla_{\bar\jmath_2}f_\alpha)(\nabla_{\bar\imath_3}\nabla_{\bar\jmath_3}f_\beta)+\ldots,
\end{aligned}
\tag{A.19}
$$

where the dots indicates terms which properly antisymmetrize F^0 in the $\bar\imath$'s and $\bar\jmath$'s. Notice also that we have removed the e^{-K} factor from the string amplitudes to adhere to the usual topological normalization, as explained in section 2. Now we can compute $\nabla_{[\bar\imath}F^0_{\bar\imath_1\bar\imath_2\bar\imath_3];\bar\jmath_1\bar\jmath_2\bar\jmath_3}$. We see then that the only term which survives is the one where $\nabla_{\bar\imath}$ acts on $f^\alpha_{\bar p}$ in the first term of (A.19) and this gives the recursion relation (3.10) for the case $g=0$ and $n=3$. The last term in (A.19) is trivially closed, whereas the other terms cancel against those arising from the commutator of D_i and $\nabla_{\bar\imath}$.

One can follow the same procedure to compute the "gravitino mass" beyond the leading order (A.5), that is the connected amplitude $\langle(\bar\chi)^6(\bar\psi)^2\rangle$. However in this case it turns out that in addition to a contribution which can be identified with $F^0_{\bar\imath_1\bar\imath_2\bar\imath_3;\bar\jmath_1\bar\jmath_2\bar\jmath_3}$, one finds also terms of different kinematic structure, involving ratios p^2/q^2, where p and q denote generically some intermediate momenta, whose interpretation is not clear.

Appendix B

In this appendix we will work out an example illustrating how the standard one-loop holomorphic anomaly of gauge couplings feeds into holomorphic anomalies associated with higher-weight interactions. More specifically, we will show that a tree-level Π^2 term discussed in Appendix A induces, in the presence of one-loop threshold corrections of the form F^1W^2, a one-loop ΠW^2 term with a coefficient $F^1_{\bar\imath;\bar\jmath}$ consistent with the recursion relations.

In order to determine $F^1_{\bar\imath;\bar\jmath}$, it is convenient to consider the amplitude $\langle z^{\bar\imath}z^{\bar\jmath}\lambda\lambda\rangle$ to second order in the external momenta corresponding to the interaction $\partial_\mu z^{\bar\imath}\partial_\mu z^{\bar\jmath}\lambda\lambda$. We will follow the same procedure as in the previous Appendix by introducing the Lagrange multipliers S^α to describe the Π^2 interaction. The one loop threshold corrections contain an

anomalous, non-holomorphic part which depends on both $\bar{Z}$ and $\bar{S}$. It corresponds to a non-local Lagrangian term, however all interaction vertices involving three or more particles can be obtained from the standard supergravity Lagrangian by taking the gauge function $H(S,Z,\bar{S},\bar{Z})$ with the anti-chiral fields identified as scalars, i.e. $\bar{Z}=\bar{z}$, $\bar{S}=\bar{s}$.

In the presence of a gauge function $H\equiv F^1+S_\alpha H_1^\alpha+\mathcal{O}(S^2)$, the zeroth order equations of motion for the Lagrange multipliers yield

$$s_\alpha^{(0)}=e^{-K}c_{\alpha\beta}\lambda\lambda(\tilde{G}_{s_\beta\bar{s}_\gamma}G^{\bar{s}_\gamma i}\partial_i F^1+H_1^\beta)+\ldots \tag{B.1}$$

where the matrix $\tilde{G}_{s_\beta\bar{s}_\gamma}$ is obtained by inverting the 2×2 block $G^{s_\beta\bar{s}_\gamma}$ of the inverse Kähler metric. It is straightforward to show the identity:

$$\tilde{G}_{s_\beta\bar{s}_\gamma}G^{\bar{s}_\gamma i}=G^{i\bar{\imath}}\partial_{\bar{\imath}}f^\beta\ , \tag{B.2}$$

which is valid to lowest order in s. In equation (B.1), we neglected terms that do not contribute to the amplitude under consideration.

After substituting the solution (B.1) into the kinetic energy part of the Lagrangian, one finds the vertex:

$$(K_{s\bar{\imath}\bar{\jmath}}\partial_\mu z^{\bar{\imath}}\partial_\mu z^{\bar{\jmath}}+K_{s\bar{k}}\partial^2 z^{\bar{k}})s_\alpha^{(0)}\ . \tag{B.3}$$

The first term contributes directly to the irreducible part of the amplitude $\langle z^{\bar{\imath}}z^{\bar{\jmath}}\lambda\lambda\rangle$, while the second term gives rise to a reducible diagram with the scalar $z^{\bar{k}}$ propagating to a 3-point vertex from the kinetic terms, of the form $K_{k\bar{\imath}\bar{\jmath}}z^k\partial^2(z^{\bar{\imath}}z^{\bar{\jmath}})$. As usual, the reducible diagram covariantizes the derivatives acting on f^α's so that the final result becomes:

$$F^1_{\bar{\imath};\bar{\jmath}}=(\nabla_{\bar{\imath}}\nabla_{\bar{\jmath}}f^\alpha)c_{\alpha\beta}[G^{k\bar{k}}(\partial_{\bar{k}}f^\beta)(\partial_k F^1)+H_1^\beta]\ , \tag{B.4}$$

where we removed the e^{-K} factor as in Appendix A.

The expression (B.4) for $F^1_{\bar{\imath};\bar{\jmath}}$ satisfies the identity (3.7), $F^1_{[\bar{\imath};\bar{\jmath}]}=0$, in a trivial way. It also satisfies the recursion relation (3.10),

$$\nabla_{[\bar{k}}F^1_{\bar{\imath}];\bar{\jmath}}=F^0_{[\bar{k}\bar{\imath}];\bar{\jmath}\bar{l}}G^{l\bar{l}}\partial_l F^1\ . \tag{B.5}$$

After taking covariant derivative $\nabla_{\bar{k}}$ of (B.4) and antisymmetrizing $\bar{\imath}$ and $\bar{k}$ indices, one finds three terms when the derivative acts inside the bracket. One of them gives the r.h.s. of eq.(B.5) while the other two are:

$$(\nabla_{\bar{\jmath}}\nabla_{[\bar{\imath}}f^{\alpha})c_{\alpha\beta}[G^{l\bar{l}}(\partial_{\bar{l}}f^{\beta})(\partial_{\bar{k}]}\partial_l F^1) + \partial_{\bar{k}]}H_1^{\beta}] \ . \tag{B.6}$$

These terms can be evaluated using the usual one loop holomorphic anomaly equation

$$\partial_A\partial_{\bar{B}}H = bG_{A\bar{B}} \ , \tag{B.7}$$

which is valid for gauge groups without charged matter, with b proportional to the beta-function. In our case, A and $\bar{B}$ denote both z moduli and auxiliary s fields. Using $\partial_{\bar{k}}H_1^{\beta} = G_{s_\beta\bar{k}} = -\partial_{\bar{k}}f^{\beta}$, one sees that the two terms in eq.(B.6) cancel against each other, which completes the verification of the recursion relation (B.5).

It is worth mentioning that the result (B.4) for $F^1_{\bar{\imath};\bar{\jmath}}$ can also be obtained from the gravitino “mass” (A.3) by using the lowest order expressions (A.4), (B.1).

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